Advanced Mathematics

Senior High

Grade 12

Teacher Guide

Standards-Based



Department of Education

'FREE ISSUE
NOT FOR SALE'

Advanced Mathematics

Senior High

Grade 12

Teacher Guide

Standards-Based



Department of Education

Issued free to schools by the Department of Education

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Subject Advisory Committee (SAC) and Board of Studies (BOS) are acknowledged for their recommendations and endorsements of this Teacher Guide.

Acronyms

AAL	Assessment As Learning
AFL	Assessment For Learning
AOL	Assessment of Learning
SSBoS	Secondary School Board of Studies
CDD	Curriculum Development Division
CRS	Classroom Response System
DA	Diagnostic Assessment
HOD	Head of Department
IHD	Integral Human Development
MTDG	Medium Term Development Goal
NGO	Non-Government Organisations
PBA	Performance Based Assessments
PNG	Papua New Guinea
SAC	Subject Advisory Committee
SBC	Standards Based Curriculum
SBE	Standards Based Education
SCG	Subject Curriculum Group
STEAM	Science, Technology, Engineering, Arts and Mathematics
STEM	Science, Technology, Engineering, and Mathematics

Secretary's Message

The aims and goals of SBC is to identify the important knowledge, skills, values and attitudes that all students are expected to acquire and master in order to effectively function in society and actively function in society and actively contribute to its development, student's welfare and enable them to acquire and apply 21st Century.

The aims of teaching and learning mathematics are to encourage and enable students to recognise that mathematics permeates the world around us. Students should be encouraged to appreciate the usefulness, power and beauty of mathematics and become confident in using mathematics to analyse and solve problems both in school and in real-life situations.

A variety of teaching and learning activities provides students with ideas to motivate students to learn, and make learning relevant, interesting and enjoyable. Teachers should provide students opportunity to develop mathematical curiosity and use inductive and deductive reasoning when solving problems and develop the knowledge, skills and attitudes necessary to pursue further studies in Mathematics.

Learning Mathematics enable students to develop abstract, logical and critical thinking and the ability to reflect critically upon their work and the work of others, develop a critical appreciation of the use of information and communication technology in mathematics appreciate the international dimension of mathematics and its multicultural and historical perspectives.

Teachers are encouraged to integrate Mathematics activities with other subjects, where appropriate, so that students can see the interrelationship between subjects and that the course they are studying provides a holistic education and a pathway for the future.

I commend and approve this Grade 12 Advanced Mathematics Teacher Guide to be used in all Senior High Schools throughout Papua New Guinea.



……..

UKE W. KOMBRA, (PhD)
Secretary for Education

Introduction

The aims of teaching and learning mathematics are to encourage and enable students to recognise that mathematics permeates the world around us. Students should be encouraged to appreciate the usefulness, power and beauty of mathematics and become confident in using mathematics to analyse and solve problems both in school and in real-life situations.

The curriculum is designed to ensure that students build a solid foundation in mathematics by connecting and applying mathematical concepts in a variety of ways and situations. To support this process, teachers should provide students opportunity to develop mathematical curiosity and use inductive and deductive reasoning when solving problems and develop the knowledge, skills and attitudes necessary to pursue further studies in mathematics.

Mathematics aims to provide a meaningful pedagogical framework for teaching and learning essential and in demand knowledge, skills, values, and attitudes that are required for the preparation of students for careers, higher education and citizenship in the 21st Century.

Students should be prepared to gather and understand information, analyse issues critically, learn independently or collaboratively, organize and communicate information, draw and justify conclusions, create new knowledge, and act ethically.

Students' employability will be enhanced through the study and application of STEAM principles. STEAM is an integral component of the core curriculum. All students are expected to study STEAM and use STEAM related skills to solve problems relating to both the natural and the physical environments. The aim of STEAM education is to create a STEAM literate society. It is envisioned that the study of STEAM will motivate students to pursue and take up academic programs and careers in STEAM related fields. STEAM has been embedded in the Mathematics curriculum. Equal opportunities should be provided for all students to learn, apply and master STEAM principles and skills.

Time allocation for Advanced Mathematics is **400** minutes for Grade 12.

Structure of the Teacher Guide

There are four main components to this teacher guide. They provide essential information on what all teachers should know and do to effectively implement the Mathematics curriculum.

Part 1 provides generic information to help the teachers to effectively use the teacher guide and the syllabus to plan, teach and assess students' performance and proficiency on the national content standards and grade-level benchmarks. The purpose of the teacher guide, syllabus and teacher guide alignment, and the four pillars of PNG SBC, which are, morals and values education, cognitive and high level thinking, and 21st Century thinking skills, STEAM, and core curriculum. These are explained to inform as well as guide the teachers so that they align SBE/SBC aims and goals, overarching and SBC principles, content standards, grade-level benchmarks, learning objectives and best practice when planning lessons, teaching, and assessing students.

Part 2 provides information on the strands, units, topics and learning objectives. How topics and learning objectives are derived is explained to the teachers to guide them to use the learning objectives provided for planning, instruction and assessment. Teachers are encouraged to develop additional topics and learning objectives to meet the learning needs of their students and communities where necessary.

Part 3 provides information on SBC planning to help guide the teachers when planning SBC lessons. Elements and standards of SBC lesson plans are described as well as how to plan for underachievers, use evidence to plan lessons, and use differentiated instruction, amongst other teaching and learning strategies.

Part 4 provides information on standards-based assessment, inclusive of performance assessment and standards, standards-based evaluation, standards-based reporting, and standards-based monitoring. This information should help the teachers to effectively assess, evaluate, report and monitor demonstration of significant aspects of a benchmark.

The above components are linked and closely aligned. They should be connected to ensure that the intended learning outcomes and the expected quality of education standards are achieved. The close alignment of planning, instruction and assessment is to the attainment of learning standards.

Purpose of the Teacher Guide

This teacher guide describes what all teachers should know and do to effectively plan, teach, and assess the Grade 12 Advanced Mathematics content to attain the required learning and proficiency standards. The overarching purpose of this teacher guide is to help teachers to effectively plan, teach, assess, evaluate, report and monitor students' learning and mastery of national and grade-level expectations.
That is, the essential knowledge, skills, values and attitudes (KSVAs) described in the content standards and grade-level benchmarks, and their achievement of the national and grade-level proficiency standards.

Ample information with thorough guidelines is provided for the teacher.

Thus, the teacher is expected to;

- understand the significance of aligning all the elements of Standards-Based Curriculum (SBC) as the basis for achieving the expected level of education quality,
- effectively align all the components of SBC when planning, teaching, and assessing students' learning and levels of proficiency,
- effectively translate and align the Biology syllabi and teacher guide to plan, teach and assess different Biology units and topics, and the KSVAs described in the grade-level benchmarks,
- understand the Biology national content standards, grade-level benchmarks, and evidence outcomes,
- effectively make sense of the content (KSVAs) described in the Biology national content standards and the essential components of the content de scribed in the grade-level benchmarks,
- effectively guide students to progressively learn and demonstrate proficiency on a range of Scientific skills, processes, concepts, ideas, principles, practices, values and attitudes,
- confidently interpret, translate and use Biology content standards and benchmark to determine the learning objectives and performance standards, and plan appropriately to enable all students to achieve these standards,
- embed the core curriculum in their Biology lesson planning, instruction, and assessment to permit all students to learn and master the core KSVAs required of all students,
- provide opportunities for all students to understand how STEAM has and continues to shape the social, political, economic, cultural, and environment contexts and the consequences, and use STEAM principles, skills, processes, ideas and concepts to inquire into and solve problems relating to both the natural and physical (man-made) worlds as well as problems created by STEAM,
- integrate cognitive skills (critical, creative, reasoning, decision-making, and problem-solving skills), high level thinking skills (analysis, synthesis and evaluation skills), values (personal, social, work, health, peace, relationship, sustaining values), and attitudes in lesson planning, instruction and assessment,

- meaningfully connect what students learn in Biology with what is learnt in other subjects to add value and enhance students' learning so that they can integrate what they learn and develop in-depth vertical and horizontal understanding of subject content,

- formulate effective SBC lesson plans using learning objectives identified for each of the topics,

- employ SBC assessment approaches to develop performance assessments to assess students' proficiency on a content standard or a component of the content standard described in the grade-level benchmark,

- effectively score and evaluate students' performance in relation to a core set of learning standards or criteria, and make sense of the data to ascertain students' expected proficiency status of progress towards me standards, and use evidence from the assessment of students' performance to develop effective evidence-based intervention strategies to help students' making inadequate or slow progress towards meeting the grade-level and national expectations to improve their learning and performance.

How to use the Teacher Guide

Teacher Guide provides essential information about what the teacher needs to know and do to effectively plan, teach and assess students learning and proficiency on learning and performance standards. The different components of the teacher guide are closely aligned with SBC principles and practice, and all the other components of PNG SBC. It should be read in conjunction with the syllabus in order to understand what is expected of teachers and students to achieve the envisaged quality of education outcomes.

The first thing teachers should do is to read and understand each of the sections of the teacher guide to help them understand the key SBC concepts and ideas, alignment of PNG SBC components, alignment of the syllabus and teacher guide, setting of content standards and grade-level benchmarks, core curriculum, STEAM, curriculum integration, essential knowledge, skills, values and attitudes, strands, units and topics, learning objectives, SBC lesson planning, and SBC assessment. A thorough understanding of these components will help teachers meet the teacher expectations for implementing the SBC curriculum, and therefore the effective implementation of the Grade 12 Mathematics Curriculum. Based on this understanding, teachers should be able to effectively use the teacher guide to do the following

Identifying Topics from Benchmarks

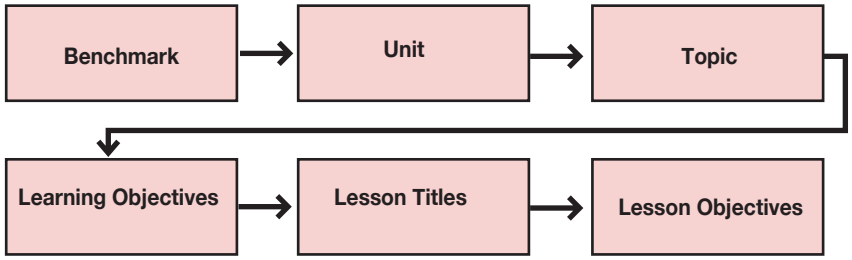
In order to identify the topic from the benchmark, the benchmark needs to be unpack. When unpacking a benchmark, identify what students will be able to know and do in order to mastered the benchmark.

Below is a description of how topics and learning objectives were derived from the grade-level benchmarks.

1. Write out the benchmark that you want to unpack.
2. Write the verbs (skills/actions) – Higher order thinking skills.
3. Underline or highlight the big idea (content) in the benchmark. The big idea (content) is the topic derived from the benchmark.
4. Write essential questions that would be engaging for students.
5. Develop sub-topics from the big idea (topic).
6. Write learning objectives according to the sub-topics.
7. Write lesson topics from the learning objectives.

Determine Lesson Objectives and Lesson Titles

Topics and learning objectives have been identified and described in the Teacher Guide. Learning objectives are derived from topics that are extracted from the grade-level benchmarks. Lesson titles are deduced from the learning objectives. Teachers should familiarise themselves with this process as it is essential for lesson planning, instruction and assessment. However, depending on the context and students' learning abilities, teachers would be required to determine additional lesson objectives and lesson titles. Teachers should use the examples provided in this teacher guide to formulate additional lesson objectives and lesson titles to meet the educational or learning needs of their students.



Identify and Teach Grade Appropriate Content

Grade appropriate content has been identified and scoped and sequenced using appropriate content organisation principles. The content is sequenced using the spiraling sequence principles. This sequencing of content will enable students to progressively learn the essential knowledge, skills, values and attitudes as they progress further into their schooling. What students learn in previous grades is reinforced and deepens in scope with an increase in the level of complexity and difficulty in the content and learning activities. It is important to understand how the content is organised so that grade appropriate content and learning activities can be selected, if not already embedded in the benchmarks and learning objectives, to not only help students learn and master the content, but ensure that what is taught is rigorous, challenging, and comparable.

Integrate the Core Curriculum in Lesson Planning, Instruction and Assessment

Teachers should use this teacher guide to help them integrate the core curriculum – values, cognitive and high level skills, 21st Century skills, STEAM principles and skills, and reading, writing, and communication skills in their lesson planning, instruction and assessment. All students in all subjects are required to learn and master these skills progressively through the education system.

Integrate Cognitive, High Level, and 21st Century Skills in Lesson Planning, Instruction and Assessment

Teachers should integrate the cognitive, high level and 21st Century skills in their annual teaching programs, and give prominence to these skills in their lesson preparation,teaching and learning activities, performance assessment, and performance standards for measuring students' proficiency on these skills.
Mathematics addresses the skills and processes of solving problems arising in everyday life, society and the workplace. Thus,students will be able to make informed decisions, problem-solving and management knowledge, skills, values and attitudes in Mathematics. This enables them to function effectively in the work and higher education environments as productive and useful citizens of a culturally diverse and democratic society in an interdependent world.

In addition, it envisaged all students attaining expected proficiency levels in these skills and will be ready to pursue careers and higher education academic programs that demand these skills, and use them in their everyday life after they leave school at the end of Grade 12. Teachers should use the teacher guide to help them to effectively embed these skills, particularly in their lesson planning and in the teaching and learning activities as well as in the assessment of students' application of the skills.

Ingrate Mathematics Values and Attitudes in Lesson Planning, Instruction and Assessment

In Mathematics, students are expected to learn, promote and use work, relationship, peace, health, social, personal, family, community, national and global values in the work and study environments as well as in their conduct as community, national and global citizens. Teachers should draw from the information and suggestions provided in the syllabus and teacher guide to integrate values and attitudes in their lesson planning, instruction, and assessment. They should report on students' progression towards internalizing different values and attitudes and provide additional support to students who are yet to reach the internalization stage to make positive progress towards this level.

Integrate Science, Technology, Engineering, Arts and Mathematics (STEAM)

Teachers should draw from both the syllabus and teacher guide in order to help them integrate STEAM principles and skills, and methodologies in their lesson planning, instruction and assessment. STEAM teaching and learning happens both inside and outside of the classroom. Effective STEAM teaching and learning requires both the teacher and the student to participate as core investigators and learners, and to work in partnership and collaboration with relevant stakeholders to achieve maximum results. Teachers should use the syllabus, teacher guides and other resources to guide them to plan and implement this and other innovative and creative approaches to STEAM teaching and learning to make STEAM principles and skills learning fun and enjoyable and, at the same time, attain the intended quality of learning outcomes.

Identify and Use Grade and Context Appropriate, Innovative, Differentiated and Creative Teaching and Learning Methodologies

SBC is an eclectic curriculum model. It is an amalgam of strengths of different curriculum types, including behavioural objectives, outcomes, and competency. Its emphasis is on students attaining clearly defined, measurable, observable and attainable learning standards, i.e., the expected level of education quality. Proficiency (competency) standards are expressed as performance standards/criteria and evidence outcomes, in real life or related situations) to indicate that they are meeting, have met or exceeded the learning standards. The selection of grade and contextually appropriate teaching and learning methodologies is critical to enabling all students to achieve the expected standard or quality of education. Teaching and learning methodologies must be aligned to the content, learning objective, and performance standard in order for the teacher to effectively teach and guide students towards meeting the performance standard for the lesson. They should be equitable and socially inclusive, differentiate, student-centred, and lifelong. They should enable STEAM principles and skills to be effectively taught and learned by students. Teachers should use the teacher guide to help them make informed decisions when selecting the types of teaching and learning methodologies to use in their teaching of the subject content, including STEAM principles and skills.

Plan Standards-Based Lessons

SBC lesson planning is quite difficult to do. However, this will be easier with more practice and experience over time. Effective SBC lesson plans must meet the required standards or criteria so that the learning objectives and performance standards are closely aligned to attain the expected learning outcomes. Teachers should use the guidelines and standards for SBC lesson planning and examples of SBC lesson plans provided in the teacher guide to plan their lessons. When planning lessons, it is important for teachers to ensure that all SBC lesson planning standards or criteria are met. If standards are not met, instruction will not lead to the attainment of intended performance and proficiency standards. Therefore, students will not attain the national content standards and grade-level benchmarks.

Use Standards-Based Assessment

Standards-Based Assessment has a number of components. These components are intertwined and serve to measure evaluate, report, and monitor students' achievement of the national and grade-level expectations, i.e., the essential knowledge, skills, values and attitudes they are expected to master and demonstrate proficiency on.

Teachers should use the information and examples on standards-based assessment assess, record, evaluate, report and monitor students' performance in relation to the learning standards.

Make informed Judgments About Students' Learning and Progress Towards Meeting Learning Standards

Teachers should use the teacher guide to effectively evaluate students' performance and use the evidence to help students to continuously improve their learning as well as their classroom practice.

It is important that teachers evaluate the performance of students in relation to the performance standards and progressively the grade-level benchmarks and content standards to make informed judgments and decisions about the quality of their work and their progress towards meeting the content standards or components of the standards. Evaluation should not focus on only one aspect of students' performance. It should aim to provide a complete picture of each student's performance. The context, inputs, processes, including teaching and learning processes, and the outcomes should be evaluated to make an informed judgment about each student's performance, Teachers should identify the causal factors for poor performance, gaps in students learning, gaps in teaching, teaching and learning resource constraints, and general attitude towards learning. Evidence-based decisions can then be made regarding the interventions for closing the gaps to allow students to make the required progress towards meeting grade-level and national expectations.

Prepare Students' Performance Reports

Reporting of students' performance and progress towards the attainment of learning standards is an essential part of SBC assessment. Results of students' performance should be communicated to particularly the students and their parents to keep them informed of students' academic achievements and learning challenges as well as what needs to be done to enable the students' make positive progress towards meeting the proficiency standards and achieve the desired level of education quality. Teachers should use the information on the reporting of students' assessment results and the templates provided to report the results of students' learning.

Monitor Students' Progress Towards Meeting the National Content Standards and Grade-Level Benchmarks

Monitoring of student's progress towards the attainment of learning standards is an essential component of standards-based assessment. It is an evidence-based process that involves the use of data from students' performance assessments to make informed judgements about students' learning and proficiency on the learning standards or their components, identify gaps in students' learning and the causal factors, set clear learning improvement targets, and develop effective evidence-based strategies (including pre-planning and re-teaching of topics), set clear time-frames, and identify measures for measuring students' progress towards achieving the learning targets.

Teachers should use the teacher guide to help them use data from students' performance assessments to identify individual students' learning weaknesses and develop interventions, in collaboration with each student and his/her parents or guardians, to address the weaknesses and monitor their progress towards meeting the agreed learning goals.

Develop additional Benchmarks

Teachers can develop additional benchmarks using the examples in the teacher guide to meet the learning needs of their students and local communities. However, these benchmarks will not be nationally assessed as these are not comparable. They are not allowed to set their own content standards or manipulate the existing ones. The setting of national content standards is done at the national level to ensure that required learning standards are maintained and monitored to sustain the required level of education quality.

Avoid Standardisation

The implementation of Grade 12 Advanced Mathematics curriculum must not be standardised. SBC does not mean that the content, lesson objectives, teaching and learning strategies, and assessment are standardised. This is a misconception and any attempt to standardise the components of curriculum without due consideration of the teaching and learning contexts, student's backgrounds and experiences, and different abilities and learning styles of students will be counterproductive. It will hinder students from achieving the expected proficiency standards and hence, high academic standards and the desired level of education quality. That is, they should not be applied across all contexts and with all students, without considering the educational needs and the characteristics of each context.

Teachers must use innovative, creative, culturally relevant, and differentiated teaching and learning approaches to teach the curriculum and enable their students to achieve the national content standards and grade-level benchmarks. And enable all students to experience success in learning the curriculum and achieve high academic standards.

What is provided in the syllabus and teacher guide are not fixed and can be changed. Teachers should use the information and examples provided in the syllabus and the teacher guide to guide them to develop, select, and use grade, context, and learner appropriate content, learning objectives, teaching and learning strategies, and performance assessment and standards. SBC is evidence-based hence decisions about the content, learning outcomes, teaching and learning strategies, students' performance, and learning interventions should be based on evidence. Teaching and learning should be continuously improved and effectively targeted using evidence from students' assessment and other sources.

Syllabus and Teacher Guide Alignment

A teacher guide is a framework that describes how to translate the content standards and benchmarks (learning standards) outlined in the syllabus into units and topics, learning objectives, lesson plans, teaching and learning strategies, performance assessment, and measures for measuring students' performance (performance standards). It expands the content overview and describes how this content identified in the content standards and their components (essential KSVAs) can be translated into meaningful and evidence-based teaching topics and learning objectives for lesson planning, instruction and assessment. It also describes and provides examples of how to evaluate and report on students' attainment of the learning standards, and use evidence from the assessment of students' performance to develop evidence-based interventions to assist students who are making slow progress towards meeting the expected proficiency levels to improve their performance.

Grade 12 Advanced Mathematics comprises of the Syllabus and Teacher Guide. These two documents are closely aligned, complimentary and mutually beneficial. They are the essential focal points for teaching and learning the essential Mathematics knowledge, skills, values and attitudes.

Syllabus and Teacher Guide Alignment

Syllabus	Teacher Guide
Outlines the ultimate aim and goals, and what to teach and why teach it	Describes how to plan, teach, and assess students' performance
• Overarching and SBC principles • Content overview • Core curriculum • Essential knowledge, skills, values and attitudes • Strands and units • Evidence outcomes • Content standards and grade-level • benchmark • Overview of assessment, evaluation, and Reporting	• Determine topics for lesson planning, instruction and assessment • Formulate learning objectives • Plan SBC lesson plans • Select teaching and learning strategies • Implement SBC assessment and evaluation • Implement SBC reporting and monitoring

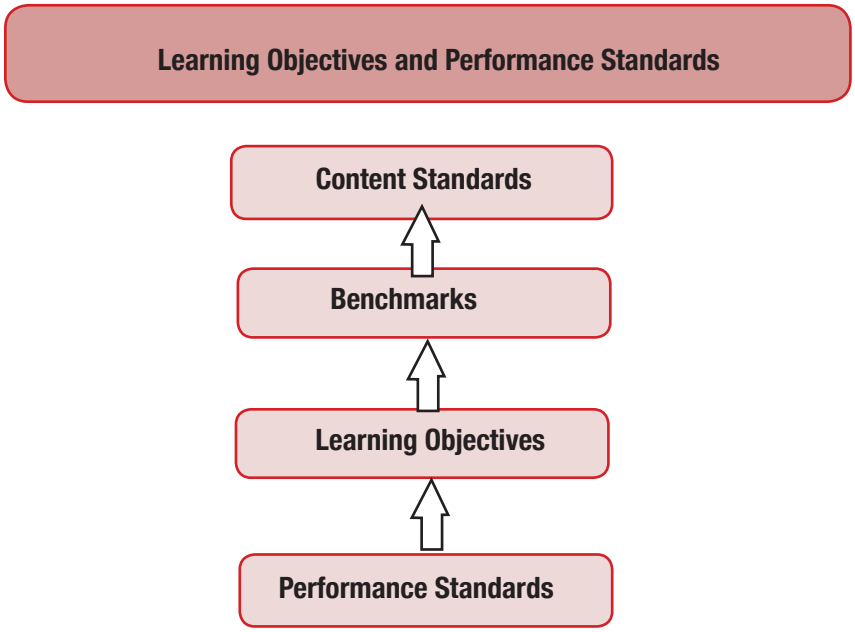
The syllabus outlines the ultimate aim and goals of SBE and SBC, what is to be taught and why it should be learned by students, the underlying principles and articulates the learning and proficiency standards that all students are expected to attain. On the other hand, the teacher guide expands on what is outlined in the syllabus by describing the approaches or the how of planning, teaching, learning, and assessing the content so that the intended learning outcomes are achieved.

This teacher guide should be used in conjunction with the syllabus. Teachers should use these documents when planning, teaching and assessing Grade 12 Advanced Mathematics content.

Teachers will extract information from the syllabus (e.g., content standards and grade-level benchmarks) for lesson planning, instruction and is for measuring students' attainment a content standard as well as progress to the next grade of schooling.

Learning and Performance Standards Alignment

Content Standards, Benchmarks, Learning Objectives, and Performance Standards are very closely linked and aligned (see below). There is a close linear relationship between these standards. Students' performance on a significant aspect of a benchmark (KSVA) is measured against a set of performance standards or criteria to determine their level of proficiency using performance assessment. Using the evidence from the performance assessment, individual student's proficiency on the aspect of the benchmark assessed and progression towards meeting the benchmark and hence the content standard are then determined.



Effective alignment of these learning standards and all the other components of PNG SBE and SBC (ultimate aim and goals, overarching, SBC and subject-based principles, core curriculum, STEAM, and cognitive, high level, and 21st Century skills) is not only critical but is also key to the achievement of high academic standards by all students and the intended level of education quality. It is essential that teachers know and can do standards alignment when planning, teaching, and assessing students' performance so that they can effectively guide their students towards meeting the grade-level benchmarks (grade expectations) and subsequently the content standards (national expectations).

Learning and Performance Standards

Standards-Based Education (SBE) and SBC are underpinned by the notion of quality. Standards define the expected level of education quality that all students should achieve at a particular point in their schooling. Students' progression and achievement of education standard (s) are measured using performance standards or criteria to determine their demonstration or performance on significant aspects of the standards and therefore their levels of proficiency or competency. When they are judged to have attained proficiency on a content standard or benchmark or components of these standards, they are then deemed to have met the standard(s) that is, achieved the intend level of education quality.

Content standards, benchmarks, and learning objectives are called learning standards while performance and proficiency standards (evidence outcomes) can be categorised as performance standards. These standards are used to measure students' performance, proficiency, progression and achievement of the desired level of education quality. Teachers are expected to understand and use these standards for lesson planning, instruction and assessment.

Content Standards

Content standards are evidence-based, rigorous and comparable regionally and globally. They have been formulated to target critical social, economic, political, cultural, environment, and employable skills gaps identified from a situational analysis. They were developed using examples and experiences from other countries and best practice, and contextualized to PNG contexts.

Content standards describe what (content - knowledge, skills, values, and attitudes) all students are expected to know and do (how well students must learn and apply what is set out in the content standards) at each grade-level before proceeding to the next grade. These standards are set at the national level and thus cannot be edited or changed by anyone except the National Subject-Based Standards Councils.

Content Standards;

- are evidenced-based,
- are rigorous and compards,
- are set at the national level,
- state or describe the expected levels of quality or achievement,
- are clear, measurable and attainable,
- are linked to and aligned with the ultimate aim and goals of SBE and SBC and overarching and SBC principles,
- delineate what matters, provide clear expectations of what students should progressively learn and achieve in school, and guide lesson planning, instruction,assessment,
- comprise knowledge, skills, values, and attitudes that are the basis for quality education,
- provide teachers a clear basis for planning, teaching, and assessing lessons, and
- provide provinces, districts, and schools with a clear focus on how to develop and organise their instruction and assessment programs as well as the content that will include in their curriculum.

Benchmarks

Benchmarks are derived from the content standards and benchmarked at the grade-level. Benchmarks are specific statements of what students should know (i.e., essential knowledge, skills, values or attitudes) at a specific grade-level or school level. They provide the basis for measuring students' attainment of a content standard as well as progress to the next grade of schooling.

Grade-level benchmarks;

- are evidenced-based,
- are rigorous and comparable to regional and global standards;
- are set at the grade level,
- are linked to the national content standards,
- are clear, measurable, observable and attainable,
- articulate grade level expectations of what students are able to demonstrate to indicate that they are making progress towards attaining the national content standards,
- provide teachers a clear basis for planning, teaching, and assessing lessons,
- state clearly what students should do with what they have learned at the end of each school-level,
- enable students' progress towards the attainment of national content standards to be measured, and
- enable PNG students' performance to be compared with the performance of PNG students with students in other countries.

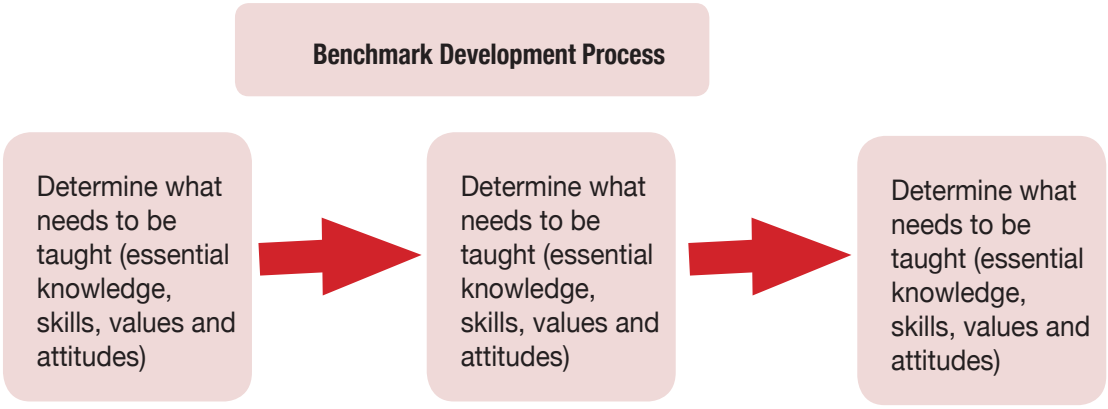
Approach for Setting National Content Standards and Grade-Level Benchmarks



Development of Additional Benchmarks

Teachers should develop additional benchmarks to meet the learning needs of their students. They should engage their students to learn about local, provincial, national and global issues that have not been catered for in the grade-level benchmarks but are important and can enhance students' understanding and application of the content. However, it is important to note that these benchmarks will not be nationally examined as they are not comparable. Only the benchmarks developed at the national level will be tested. This does not mean that teachers should not develop additional benchmarks. An innovative, reflect, creative and reflexive teacher will continuously reflect on his/her classroom practice and use evidence to provide challenging, relevant, and enjoyable learning opportunities for his/her students to build on the national expectations for students.

Teachers should follow the following process when developing additional grade-level benchmarks.



Learning Objectives

Learning or instructional Objectives are precise statements of educational intent. They are formulated using a significant aspect or a topic derived from the benchmark, and is aligned with the educational goals, content standards, benchmarks, and performance standards. Learning objectives are stated in outcomes language that describes the products or behaviours that will be provided by students. They are stated in terms of measurable and observable student behaviour. For example, students will be able to explore the idea of direct and indirect proportion.

Performance Standards

Performance Standards are concrete statements of how well students must learn what is set out in the content standards, often called the "be able to do" of "what students should know and be able to do." Performance standards are the indicators of quality that specify how competent a students' demonstration or performance must be. They are explicit definitions of what students must do to demonstrate proficiency or competency at a specific level on the content standards.

Performance standards:

- measure students' performance and proficiency **(using performance indicators)** in the use of a specific knowledge, skill, value, or attitude in real life or related situations.
- provide the basis **(performance indicators)** for evaluating, reporting and monitoring students' level of proficiency in use of a specific knowledge, skills, value, or attitude.
- are used to plan for individual instruction to help students not yet meeting expectations **(desired level of mastery and proficiency)** to make adequate progress towards the full attainment of benchmarks and content standards.
- are used as the basis for measuring students' progress towards meeting grade-level benchmarks and content standards.

Proficiency Standards

Proficiency standards describe what all students in a particular grade or school level can do at the end of a strand, or unit. These standards are sometimes called evidence outcomes because they indicate if students can actually apply or use what they have learned in real life or similar situations. They are also categorized as benchmarks because that is what all students are expected to do before exiting a grade or are deemed ready for the next grade.

Core Curriculum

A core set of common learnings (knowledge, skills, values, and attitudes) are integrated into the content standards and grade-level benchmarks for all subjects. This is to equip all students with the most essential and in-demand knowledge, skills, and dispositions they will need to be successful in modern/postmodern work places, higher-education programs and to be productive, responsible, considerate, and harmonious citizens. Common set of learning are spirally sequenced from Preparatory-Grade 12 to deepen the scope and increase the level of difficulty in the learning activities so that what is learned is reinforced at different grade levels.

The core curriculum includes:

- cognitive (thinking) skills (refer to the syllabus for a list of these skills),
- reasoning, decision-making and problem-solving skills,
- high level thinking skills (analysis, synthesis and evaluation skills),
- 21st Century skills (refer to illustrative list in the appendix 2),
- reading, writing and communication skills,
- STEAM principles and skills,
- essential values and attitudes (core personal and social values, and sustaining values), and
- spiritual values and virtues.

The essential knowledge, skills, values and attitudes comprising the core curriculum are interwoven and provide an essential and holistic framework for preparing all students for careers, higher education and citizenship.

All teachers are expected to include the core learnings in their lesson planning, teaching, and assessment of students in all their lessons. They are expected to foster, promote and model the essential values and attitudes as well as the spiritual values and virtues in their conduct, practice, appearance, and their relationships and in their professional and personal lives. In addition, teachers are expected to mentor, mould and shape each student to evolve and possess the qualities envisioned by society.

Core values and attitudes must not be taught in the classroom only; they must also be demonstrated by students in real life or related situations inside and outside of the classroom, at home, and in everyday life. Likewise, they must be promoted, fostered and modelled by the school community and its stakeholders, especially parents. A holistic of school approach to values and attitudes in teaching, promoting and modelling is critical to students and the whole school community to internalise the core values and attitudes and make them habitual in their work and school place, and in everyday life. Be it work values, relationship values, peace values, health values, personal and social values, or religious values, teachers should give equal prominence to all common learnings in their lesson planning, teaching, assessment, and learning interventions. Common learnings must be at the heart of all teaching and extra-curricular programs and activities.

Science, Technology, Engineering, Arts and Mathematics

STEAM education is an integrated, multidisciplinary approach to learning that uses science,technology, engineering, arts and mathematics as the basis for inquiring about how STEAM has and continues to change and impact the social, political, economic, cultural and environmental contexts and identifying and solving authentic (real life) natural and physical environment problems by integrating STEAM-based principles, cognitive, high level and 21st Century skills and processes, and values and attitudes.

Mathematics is focused on both goals of STEAM rather than just the goal of problem-solving. This is to ensure that all students are provided opportunities to learn, integrate, and demonstrate proficiency on all essential STEAM principles, processes, skills, values and attitudes to prepare them for careers, higher education and citizenship.

Through STEAM education students will be able to:

(i) examine and use evidence to draw conclusions about how STEAM has and continues to change the social, political, economic, cultural and environmental contexts.

(ii) Investigate and draw conclusions on the impact of STEAM solutions to problems on the social, political, economic, cultural and environmental contexts.

(iii) Identify and solve problems using STEAM principles, skills, concepts, ideas and process.

(iv) Identify, analyse and select the best solution to address a problem.

(v) build prototypes or models of solutions to problems.

(vi) replicate a problem solution by building models and explaining how the problem was or could be solved.

(vii) test and reflect on the best solution chosen to solve a problem.

(viii) collaborate with others on a problem and provide a report on the process of problem solving used to solve the problem.

(ix) use skills and processes learnt from lessons to work on and complete STEAM projects.

(x) demonstrate STEAM principles, skills, processes, concepts and ideas through simulation and modelling.

(xi) explain the significance of values and attitudes in problem-solving.

STEAM is a multidisciplinary and integrated approach to understanding how science, technology, engineering, arts and mathematics shape and are shaped by our material, intellectual, cultural, economic, social, political and environmental contexts. And for teaching students the essential in demand cognitive, high level and 21st Century skills, values and attitudes, and empower them to effectively use these skills and predispositions to identify and solve problems relating to the natural and physical environments as well as the impact of STEAM-based solutions on human existence and livelihoods, and on the social, political, economic, cultural, and environmental systems.

STEAM disciplines have and continue to shape the way we perceive knowledge and reality, think and act, our values, attitudes, and behaviours, and the way we relate to each other and the environment. Most of the things we enjoy and consume are developed using STEAM principles, skills, process, concepts and ideas.

Things humans used and enjoyed in the past and at present are developed by scientists, technologists, engineers, artists and mathematicians to address particular human needs and wants. Overtime, more needs were identified and more products were developed to meet the ever changing and evolving human needs. What is produced and used is continuously reflected upon, evaluated, redesigned, and improved to make it more advanced, multipurpose, fit for purpose, and targeted towards not only improving the prevailing social, political, economic, cultural and environmental conditions but also to effectively respond to the evolving and changing dynamics of human needs and wants. And, at the same time, solutions to human problems and needs are being investigated and designed to address problems that are yet to be addressed and concurred. This is an evolving and ongoing problem-solving process that integrates cognitive, high level, and 21st Century skills, and appropriate values and attitudes.

STEAM is a significant framework and focal point for teaching and guiding students to learn, master and use a broad range of skills and processes required to meet the skills demands of PNG and the 21st Century. The skills that students will learn will reflect the demands that will be placed upon them in a complex, competitive, knowledge-based, information-age,technology- driven economy and society. These skills include cognitive (critical, synthetic, creative, reasoning, decision-making, and problem-solving) skills, high level (analysis, synthesis and evaluation) skills and 21st Century skills (see Appendix 4). Knowledge-based, information, and technology driven economies require knowledge workers not technicians. Knowledge workers are lifelong learners, are problem solvers, innovators, creators, critical and creative thinkers, reflective practitioners, researchers (knowledge producers rather than knowledge consumers), solutions seekers,outcomes oriented, evidence-based decision makers, and enablers of improved and better outcomes for all.

STEAM focuses on the skills and processes of problem solving. These skills and processes are at the heart of the STEAM movement and approach to not only problem solving and providing evidence-based solutions but also the development and use of other essential cognitive, high level and 21st Century skills. These skills are intertwined and used simultaneously to gain a broader understanding of the problems to enable creative, innovative, contextually relevant, and best solutions to be developed and implemented to solve the problems and attain the desired outcomes. It is assumed that by teaching students STEAM-based problem-solving skills and providing learning opportunities inside and outside the classroom will motivate more of them to pursue careers and academic programs in STEAM related fields thus, closing the skills gaps and providing a pool of cadre of workers required by technology, engineering, science, and mathematics-oriented industries.

Although, STEAM focuses on the development and application of skills in authentic (real life) contexts, for example the use of problem- solving skills to identify and solve problems relating to the natural and physical worlds, it does not take into account the significant influence values and attitudes have on the entire process of problem solving. Values and attitudes are intertwined with knowledge and skills. Knowledge, skills, values and attitudes are inseparable. Decisions about skills and processes of skills development and application are influenced by values and attitudes (mindset) that people hold. In the same light, the use of STEAM principles, processes and skills to solve problems in order to achieve the outcomes envisaged by society are influenced by values and the mindset of those who have identified and investigated the problem as well as those who are affected by the problem and will benefit from the outcome.

STEAM Problem-Solving Methods and Approaches

Problem-solving involves the use of problem-solving methods and processes to identify and define a problem, gather information to understand its causes, draw conclusions, and use the evidence to design and implement solutions to address it. Even though there are many different problem-solving methods and approaches, they share some of the steps of problem-solving, such as;

- identifying the problem,
- understanding the problem by collecting data,
- analyse and interpret the data,
- draw conclusions,
- use data to consider possible solutions,
- select the best solution,
- test the effectiveness of the solution by trialling and evaluating it, and
- review and improve the solution.

STEAM problem solving processes go from simple and technical to advance and knowledge-based processes. However, regardless of the type of process used, students should be provided opportunities to learn the essential principles and processes of problem solving and, more significantly, to design and create a product that addressed a real problem and meets a human need.

The following are some of the STEAM problem solving processes.

1. Engineering and Technology Problem Solving Methods and Approaches

Engineering and technology problem-solving methods are used to identify and solve problems relating to the physical world using the design process. The following are some of the methods and approaches used to solve engineering and technology related problems.

Parts Substitution
It is the most basic of the problem-solving methods. It simply requires the parts to be substituted until the problem is solved.

Diagnostics
After identifying a problem, the technician would run tests to pinpoint the fault. The test results would be used either as a guide for further testing or for replacement of a part, which also need to be tested. This process continues until the solution is found and the device is operating properly.

Troubleshooting
Troubleshooting is a form of problem solving, often applied to repair failed products or processes.

Reverse Engineering
Reverse engineering is the process of discovering the technological principles underlying the design of a device by taking the device apart, or carefully tracing its workings or its circuitry. It is useful when students are attempting to build something for which they have no formal drawings or schematics.

Divide and Conquer

Divide and conquer is the technique of breaking down a problem into sub-problems, then breaking the sub-problems down even further until each of them is simple enough to be solved. Divide and conquer may be applied to all groups of students to tackle sub-problems of a larger problem, or when a problem is so large that its solution cannot be visualised without breaking it down into smaller components.

Extreme Cases

Considering "extreme cases" – envisioning the problem in a greatly exaggerated or greatly simplified form, or testing using extreme condition – can often help to pinpoint a problem. An example of the extreme-case method is purposely inputting an extremely high number to test a computer program.

Trial and Error

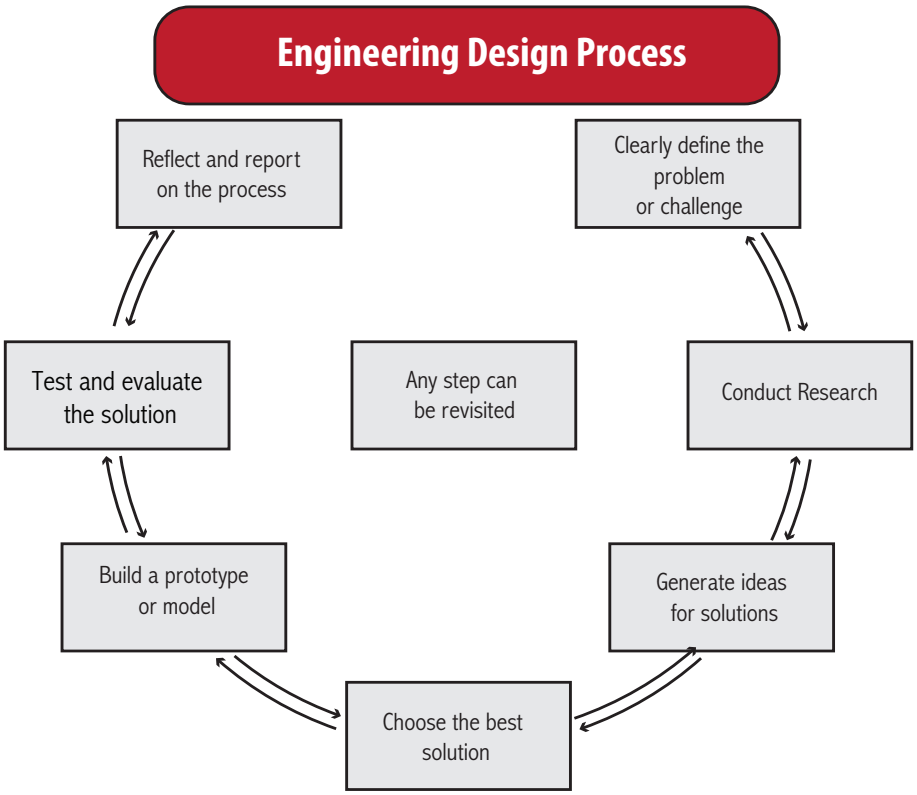
The trial and error method involve trying different approaches until a solution is found. It is often used as a last resort when other methods have been exhausted.

- Test and evaluate the solution.
- Repeat steps as necessary to modify the design or correct faults.
- Reflect and report on the process.

1. Engineering Design Process

Technological fields use the engineering design process to identify and define the problem or challenge, investigate the problem, collect and analyse data, and use the data to formulate potential solutions to the problem, analyse each of the solutions in terms of its strengths and weaknesses, and choose the best solution to solve the problem. It is an open-ended problem-solving process that involves the full planning and development of products or services to meet identified needs. It involves a sequence of steps as illustrated.

1) Analyse the context and background, and clearly define the problem.
2) Conduct research to determine design criteria, financial or other constraints, and availability of materials.
3) Generate ideas for potential solutions, using processes such as brainstorming and sketching.
4) Choose the best solution.
5) Build a prototype or model.
6) Test and evaluate the solution.
7) Repeat steps as necessary to modify the design or correct faults.
8) Reflect and report on the process.



STEAM-Based Lesson planning

Effective STEAM lesson planning is key to the achievement of expected STEAM outcomes. STEAM skills can be planed and taught using separate STEAM-based lesson plans or integrated into the standards-based lesson plans. To effectively do this, teachers should know how to write effective standards and STEAM-based lesson plans.

An example of a STEAM-based lesson plan is provided in appendix. Teachers should use this to guide them to integrate STEAM content and teaching, learning and assessment strategies into their standards-based lesson plans.

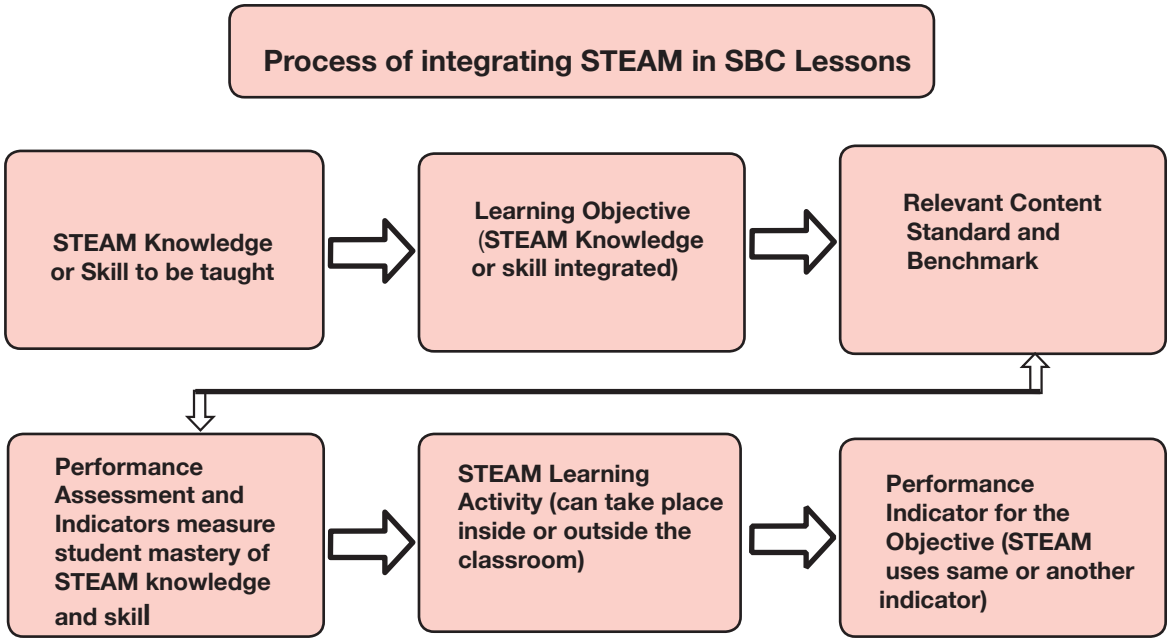
Integration of STEAM problem-solving skills into standards-based lesson plans

Knowing how to integrate STEAM problem-solving skills, principles, values and attitudes as well as STEAM teaching, learning, and assessment strategies into standards-based lesson plans is essential for achieving the desired STEAM learning outcomes. When integrating STEAM problem-solving skills into the standards-based lesson plans, teachers should ensure that these skills are not only effectively aligned to the learning objective and performance standards, they must also be effectively taught and assessed.

STEAM principles and problem-solving skills are integrated into the content standards and grade-level benchmarks. A list of these skills, including 21st Century skills, is provided in the syllabus. Teachers should ensure that these skills are integrated in their standards-based lesson plans, taught and assessed to determine students' level of proficiency on each skill or specific components of the skill.

Teachers are expected to integrate the essential STEAM principles, processes, skills, values and attitudes described in the Grade 12 benchmarks when formulating their standards-based lesson plans. Opportunities should be provided inside and outside of the classroom for students to learn, explore, model and apply what they learn in real life or related situations. These learning experiences will enable students to develop a deeper understanding of STEAM principles, processes, skills, values and attitudes and appreciate their application in real life to solve problems.

Teachers should use the following process as guide to integrate STEAM principles and problem-solving skills into the standards-based lesson plans.



Steps for integrating STEAM problem-solving principles and skills into standards-based lesson plans.

Step 1: Identify the STEAM knowledge or skill to be taught (from the table of KSVAs for each content standard and benchmark). This is captured in the learning objective stated in the standards-based lesson plan.

Step 2: Develop and include a performance standard or indicator for measuring student mastery of the STEAM knowledge or skill (e.g. level of acceptable competency or proficiency) if this is different from the one already stated in the lesson plan.

Step 3: Develop student learning activity (An activity that will provide students the opportunity to apply the STEAM knowledge or skill specified by the learning objective and appropriate statement of the standards). Activity can take place inside or outside of the classroom, and during or after school hours.

Step 4: Develop and use performance descriptors (standards or indicators) to analyse students' STEAM related behaviours and products (results or outcomes), which provide evidence that the student has acquired and mastered the knowledge or skill of the learning objective specified by the indicator (s) of the standard(s).

STEAM Teaching Strategies

STEAM education takes place in both formal and informal classroom settings. It takes place during and after school hours. It is a continuous process of inquiry, data analysis, making decisions about interventions, and implementing and monitoring interventions for improvements.

There are a variety of STEAM teaching strategies. However, teaching strategies selected must enable teachers to guide students to use the engineering and artistic design processes to identify and solve natural and physical environment problems by designing prototypes and testing and refining them to effectively mitigate the problems identified. The following are some of the strategies that could be used to utilise the STEAM approach to solve problems and coming up with technological solutions.

1. Inquiry-Based Learning
2. Problem-Based Learning
3. Project-based learning
4. Collaborative Learning

Collaborative learning involves individuals from different STEAM disciplines and expertise in a variety of STEAM problem solving approaches working together and sharing their expertise and experiences to inquire into and solve a problem. Teachers should plan to provide students opportunities to work in collaboration and partnership with experts and practitioners engaged in STEAM related careers or disciplines to learn first-hand about how STEAM related skills, processes, concepts, and ideas are applied in real life to solve problems created by natural and physical environments. Collaborative learning experiences can be provided after school or during school holidays to enable students to work with STEAM experts and practitioners to inquiry and solve problems by developing creative, innovative and sustainable solutions. Providing real life experiences and lessons, e.g., by involving students to actually solve a scientific, technological, engineering, or mathematical, or Arts problem, would probably spark their interest in a STEAM career path.

Developing STEAM partnerships with external stakeholders e.g., high education institutions, private sector, research and development institutions, and volunteer and community development organizations can enhance students' learning and application of STEAM problem solving principles and skills.

Some examples of STEAM-related partnership experiences may include:

- Participatory Learning
- Group-Based Learning
- Task Oriented Learning
- Action Learning
- Experiential Learning
- Modelling
- Simulation.

STEAM Learning Strategies

Teachers should include in their lesson plans STEAM learning activities. These activities should be aligned to principle or a skill planned for students to learn and demonstrate proficiency at the end of the lesson to expose students to STEAM and giving them opportunities to explore STEAM-related concepts, they will develop a passion for it and, hopefully, pursue a job in a STEAM field.
Providing real life experiences and lessons, e.g., by involving students to actually solve a scientific, technological, engineering, or mathematical, or arts problem, would probably spark their interest in a STEAM career path. This is the theory behind STEAM education.

STEAM-Based Assessment

STEAM-based assessment is closely linked to standards-based assessment where assessment is used to assess students' level of competency or proficiency of a specific knowledge, skill, value, or attitude taught using a set of performance standards (indicators or descriptors). The link also includes the main components such as the purpose, the assessment principles and assessment strategies and tools.
In STEAM-based assessment, assessments are designed for what students should know and be able to do. In STEAM learning, students are assessed in a variety of ways including portfolios, project/problem-based assessments, backwards design, authentic assessments, or other student-centered approaches.
When planning and designing the assessment, teachers should consider the authenticity of the assessment by designing an assessment that relates to a real world task or discipline specific attributes such as simulation, role play, placement assessment, live projects and debates. These tasks should make the activity meaningful to the student, and therefore be motivating as well as developing employability skills and discipline specific attributes.

Effective STEAM-Based Assessment Strategies

The following are the six assessment tools and strategies to impact teaching and learning as well as help teachers foster 21st Century learning environment in their classrooms.

1. Rubrics
2. Performance-Based Assessments (PBAs)
3. Portfolios
4. Student self-assessment
5. Peer-assessment
6. Student Response Systems (SRS).

Although the list does not include all innovative assessment strategies, it includes what we think are the most common strategies, and ones that may be particularly relevant to the educational context of developing countries in this 21st Century. Many of the assessment strategies currently in use fit under one or more of the categories discussed. Furthermore, it is important to note that these strategies also connect in a variety of ways.

1. Rubrics

Rubrics are both a tool to measure students' knowledge and ability as well as an assessment strategy. A rubric allows teachers to measure certain skills and abilities not measurable by standardized testing systems that assess discrete knowledge at a fixed moment in time. Rubrics are also frequently used as part of other assessment strategies including; portfolios, performances, projects, peer-review and self-assessment which are also elaborated in this section.

2. Performance-Based Assessments

Performance-Based Assessments (PBA), also known as project-based or authentic assessments, are generally used as a summative evaluation strategy to capture not only what students know about a topic, but if they have the skills to apply that knowledge in a "real-world" situation.
By asking them to create an end product. PBA pushes students to synthesize their knowledge and apply their skills to a potentially unfamiliar set of circumstances that is likely to occur beyond the confines of a controlled classroom setting.
The implementation of performance-based assessment strategies can also impact other instructional strategies in the classroom.

3. Portfolio Assessment

Portfolios are a collection of student work gathered over time that is primarily used as a summative evaluation method. The most salient characteristic of the portfolio assessment is that rather than being a snapshot of a student's knowledge at one point in time (like a single standardized test), it highlights student effort, development, and achievement over a period of time; portfolios measure a student's ability to apply knowledge rather than simply regurgitate. They are considered both student-centred and authentic assessments of learning.

4. Self-assessment

While the previous assessment tools and strategies listed in this report generally function as summative approaches, self-assessment is generally viewed as a formative strategy, rather than one used to determine a student's final grade. Its main purpose is for students to identify their own strengths and weakness and to work to make improvements to meet specific criteria.
Self-assessment occurs when students judge their own work to improve performance as they identify discrepancies between current and desired performance. In this way, self-assessment aligns well with standards- based education because it provides clear targets and specific criteria against which students or teachers can measure learning.

Self-assessment is used to promote self-regulation, to help students reflect on their progress and to inform revisions and improvements on a project or paper. In order for self-assessment to be truly effective four conditions must be in place: the self- assessment criteria is negotiated between teachers and students, students are taught how to apply the criteria, students receive feedback on their self-assessments and teachers help students use assessment data to develop an action plan.

5. Peer assessment

Peer assessment, much like self-assessment, is a formative assessment strategy that gives students a key role in evaluating learning. Peer assessment approaches can vary greatly but, essentially process develops both the assessor and assessee's skills and knowledge.

The primary goal for using peer assessment is to provide feedback to learners. This strategy may be particularly relevant in classrooms with many students per teacher since student time will always be more plentiful than teacher time. Although any single student's feedback may not be rich or in-depth as tea her's feedback, the research suggests that peer assessment can improve learning.

6. Student Response System

Student response system (SRS), also known as classroom response (CRS), audience response system (ARS) is a general term that refers to a variety of technology-based formative assessment tools that can be used to gather student-level data instantly in the classroom. Through the combination of hardware, (voice recorders, PC, internet connection, projector and screen) and software.

Teachers can ask students a wide range of questions (both closed and open ended), where students can respond quickly and anonymously, and the teacher can display the data immediately and graphically. The use of technology also includes a use of video which examines how a range of strategies can be used to assess students' understanding.

The value of SRS comes from teachers analysing information quickly and then devising real-time instructional solutions to maximize student learning. This includes a suggested approach to help teachers and trainers assess learning.

Curriculum Integration

What is Curriculum Integration?

Curriculum integration is making connections in learning across the curriculum. The ultimate aim of curriculum integration is to act as a bridge to increase students' achievement and engage in relevant curriculum.

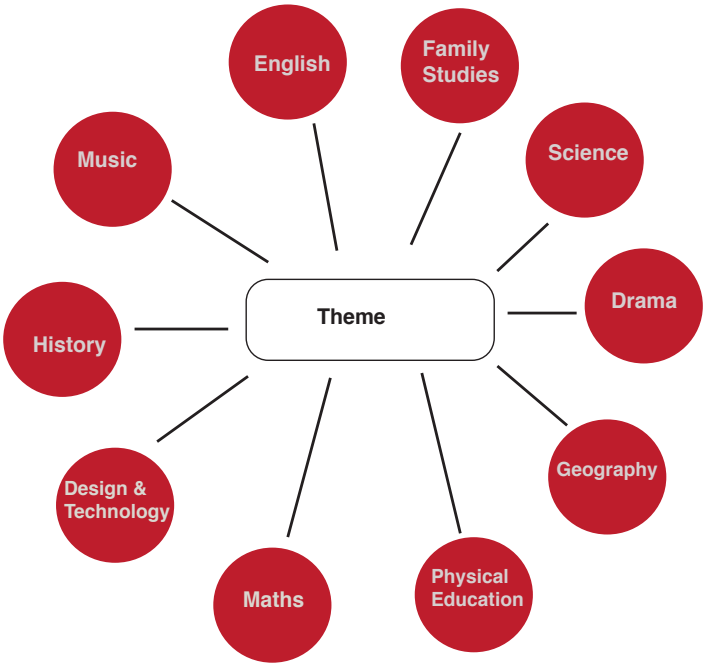
Teachers must develop intriguing curriculum by going beyond the traditional teaching of content based or fragmented teaching to one who is knowledge based and who should be perceived as a 21st Century innovative educator. Curriculum integration is a holistic approach to learning thus curriculum integration in PNG SBC will have to equip students with the essential knowledge, skills, values and attitudes that are deemed 21st Century.

There are three approaches that PNG SBC will engage to foster conducive learning for all its children whereby they all can demonstrate proficiency at any point of exit. Adapting these approaches will have an immense impact on the lives of these children thus they can be able to see themselves as catalyst of change for a competitive PNG. Not only that but they will be comparable to the world standards and as global citizens.

Engaging these three approaches in our curriculum will surely sharpen the knowledge and ability of each child who will foresee themselves as assets through their achievements thus contribute meaningfully to their country. They themselves are the agents of change. Integrated learning will bear forth a generation of knowledge based populace who can solve problems and make proper decisions based on evidence. Thus, PNG can achieve its goals like the Medium Term Development Goals (MTDG) and aims such as the Vision 2050 for a happy, healthy and wealthy society whereby, all its citizens should have access and fair distribution to income, shelter, health, education and general goods and services improving the general standard of living for PNG in the long run.

1. (i) Multidisciplinary Approach

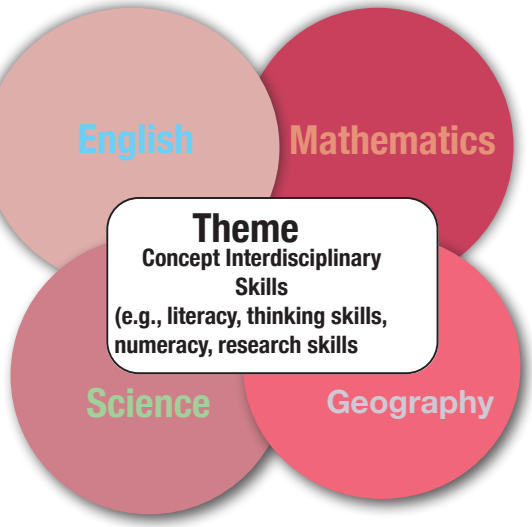
In this approach learning involves a theme or concept that will be taught right across all subject area of study by students. That is, content of a particular theme will be taught right across all subjects as shown in the diagram below. For instance, if the theme is global warming, subject areas create lessons or assessment as per their subjects around this theme. Social Science will address this issue, Science and all other subject likewise.



1. (ii) Interdisciplinary Approach

This approach addresses learning similarly to the multidisciplinary approach of integrated learning whereby learning takes place within the subject area. However, it is termed interdisciplinary in that the core curriculum of learning is interwoven into each subject under study by the students. For instance; in Social Science under the strand of geography students write essay on internal migration however, apart from addressing the issues of this topic, they are to apply the skill of writing text types in their essay such as argumentative essay, informative, explanatory, descriptive, expository and narrative essay while writing their essay. They must be able to capture the mechanics of English skills such as grammar, punctuation and so forth. Though these skills are studied under English they are considered as core skills that cut across all subjects under study. For example; if Science students were to write about human development in biology then the application of writing skills has to be captured by the students in their writing. It is not seen as an English skill but a standard essential skill all students must know and do regardless.

Therefore, essential knowledge, skills, values and attitudes comprising the core curriculum are interwoven and provide an essential and holistic framework for preparing all students for careers, higher education and citizenship in this learning.



2. Intradisciplinary Approach

This approach involves teachers integrate sub disciplines within a subject area. For instance, within the subject Social Science, the strands (disciplines) of geography, environment, history, political science and environment will all be captured studying a particular content for Social Science. For example, under global warming, students will study the geographical aspects of global warming, environmental aspect of global warming and likewise for history, political science and economics. Thus, children are well aware of the issues surrounding global warming and can address it confidently at each level of learning.

3. Trans disciplinary Approach

In this approach learning goes beyond the subject area of study. Learning is organized around students' questions and concerns. That is, where there is a need for change to improve lives, students develop their own curriculum to effect these need. The trans-disciplinary approach addresses real-life situations thus giving the opportunity to students to attain real life skills. This learning approach is more to do with Project–Based Learning also referred to as problem-based learning or place-based learning.

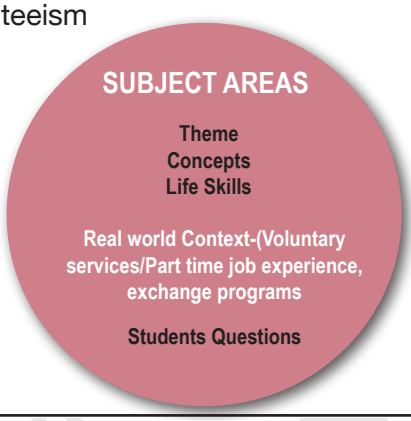
Below are the three steps to planning project based curriculum

1. Teachers and students select a topic of study based on student interests,
2. curriculum standards, and local resources.
3. The teacher finds out what the students already know and helps them generate questions to explore. The teacher also provides resources for students and
4. opportunities to work in the field
5. Students share their work with others in a culminating activity. Students display the results of their exploration and review and evaluate the project.

For instance; students may come up with slogans for school programs such as 'Our culture – clean city for a healthier PNG'. The main aim could be to curb betel nut chewing in public areas especially around bus stops and local markets. Here, students draw up their own instructions and criteria for assessment which is; they have to clean the nearest bus stop or local market once a week throughout the year. They also design and create posters to educate the general public as their program continues. They can also involve the town council and media to assist them especially to carry out awareness.

Studies have proven that Project based-programs have led to the following:

- Students go far beyond the minimum effort
- Make connections among different subject areas to answer open-ended questions
- Retain what they have learnt
- Apply learning to real-life problems
- Have fewer discipline problems
- Lower absenteeism



These integrated learning approaches will demand for teaches to be proactive in order to improve students learning and achievement. In order for PNG Standards-Based Curriculum to serve its purpose fully, these three approaches must be engaged for better learning for the children of PNG now and in the future

Essential Knowledge, Skills, Values and Attitude & Mathematical Thinking

Students' level of proficiency and progression towards the attainment of content standards will depend on their mastery and application of essential knowledge, skills, values, and attitudes in real life or related situations. Provided here are examples of different types of knowledge, processes, skills, values, and attitudes that all students are expected to learn and master as they progress through the grades. These are expanded and deepen in scope and the level of difficulty and complexity are increased to enable students to study in-depth the subject content as they progress from one grade to the next.

These knowledge, skills, values and attitudes have been integrated into the content standards and benchmarks. They will also be integrated into the performance standards. Teachers are expected to plan and teach essential knowledge, skills, values and attitudes in their lessons, and assess students' performance and proficiency, and progression towards the attainment of content standards.

Types of Knowledge

There are different types of knowledge. These include;	
• Public and private (privileged) knowledge • Specialised knowledge • Good and bad knowledge • Concepts, processes, ideas, skills, values, attitudes • Theory and practice • Fiction and non-fiction • Traditional, modern, and postmodern knowledge	• Subject and discipline-based knowledge • Lived experiences • Evidence and assumptions • Ethics and Morales • Belief systems • Facts and opinions • Wisdom • Research evidence and findings • Solutions to problems

Types of Processes

There are different types of processes. These include;	
• Problem-solving • Logical reasoning • Decision-making • Reflection	• Cyclic processes • Mapping (e.g. concept mapping) • Modelling • Simulating
Mathematics Inquiry processes include: • Gathering information • Analysing information • Evaluating information • Making judgements • Taking actions	

Mathematical Thinking Processes

The five Mathematical process skills that can help the students improve their mathematical thinking.

1. Mathematical Problem Solving

- Understand the meaning of the problem and look for entry points to its solution
- Analyse information (givens, constrains, relationships, goals)
- Make conjectures and plan a solution pathway
- Monitor and evaluate the progress and change course as necessary
- Check answers to problems and ask, “Does this make sense?

2. Mathematical Communication

- Use definitions and previously established causes/effects (results) in constructing arguments
- Make conjectures and use counter examples to build a logical progression of statements to explore and support their ideas
- Communicate and defend mathematical reasoning using objects, drawings, diagrams, actions
- Listen to or read the arguments of others
- Decide if the arguments of others make sense and ask probing questions to clarify or improve the arguments.

3. Mathematical Reasoning

- Make sense of quantities and relationships in problem situations
- Represent abstract situations symbolically and understand the meaning of quantities
- Create a coherent representation of the problem at hand
- Consider the units involved
- Flexibly use properties of operations.

4. Mathematical Connections

- Look for patterns or structure, recognizing that quantities can be represented in different ways
- Recognize the significance in concepts and models and use the patterns or structure for solving related problems
- View complicated quantities both as single objects or compositions of several objects and use operations to make sense of problems
- Notice repeated calculations and look for general methods and short cuts
- Continually evaluate the reasonableness of intermediate results (comparing estimates) while attending to details and make generalizations based on finding.

5. Mathematical Representation

- Apply prior knowledge to solve real world problems
- Identify important quantities and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas
- Make assumptions and approximations to make a problem simpler
- Check to see if an answer makes sense within the context of a situation and change a model when necessary.

Types of Skills

There are different types of skills. These include:

1. Cognitive (Thinking) Skills

Thinking skills can be categorized into **critical thinking** and **creative thinking** skills.

i. Critical Thinking Skills

A person who thinks critically always evaluates an idea in a systematic manner before accepting or rejecting it. Critical thinking skills include;	
• Attributing • Comparing and contrasting • Grouping and classifying • Sequencing • Prioritising • Analysing	• Detecting bias • Evaluating • Metacognition (Thinking about thinking) • Making informed conclusions.

ii. CreativeThinking Skills

A person who thinks creatively has a high level of imagination, able to generate original and innovative ideas, and able to modify ideas and products. Creative thinking skills include;	
• Generating ideas • Deconstruction and reconstruction • Relating • Making inferences • Predicting • Making generalisations • Visualizing	• Synthesising • Making hypothesis • Making analogies • Invention • Transformation • Modelling • Simulating

2. **Reasoning Skills** - Reason is a skill used in making a logical, just, and rational judgment.

3. **Decision-Making Skills** - Decision-making involves selection of the best solution from various alternatives based on specific criteria and evidence to achieve a specific aim.

4. **Problem Solving Skills** – These skills involve finding solutions to challenges or unfamiliar situations or unanticipated difficulties in a systematic manner.

5. Literacy Skills

A strong emphasis must be placed on various types of literacy, from financial to technological, from media to mathematical, from content to cultural. Literacy may be defined as the ability of an individual to use information to function in society, to achieve goals and to develop her or his knowledge and potential. Teachers emphasize certain aspects of literacy over others, depending on the nature of the content and skills they want students to learn.

The following literacy skills are intended to be exemplary rather than defi nitive	
• Listens, read, write, and speak with comprehension and clarity • Define and apply discipline-based conceptual vocabulary • Describe people, places, and events, and the connections between and among them • Arrange events in chronological sequence • Differentiate fact from opinion • Determine an author's purpose • Determine and analyse similarities and differences • Analyse cause and effect relationships • Explore complex patterns, interactions and relationships • Differentiate between and among various options	• Listens, read, write, and speak with comprehension and clarity • Define and apply discipline-based conceptual vocabulary • Describe people, places, and events, and the connections between and among them • Arrange events in chronological sequence • Differentiate fact from opinion • Determine an author's purpose • Determine and analyse similarities and differences • Analyse cause and effect relationships • Develop an ability to use and apply abstract principals • Explore and/or observe, identify, and analyse how individuals and/or societies relate to one another

6. **High Level Thinking Skills -** These skills include analysis, synthesis, and evaluation skills.

i. Analysis Skills – Analysis skills involve examining in detail and breaking information into parts by identifying motives or causes,underlying assumptions, hidden messages; making inferences and finding evidence to support generalisations, claims, and conclusions.

Keywords				
Analyse	Differences	Find	List	Similar to
Appraise	Discover	Focus	Motivate	Simplify
Arrange	Discriminate	Function	Omit	Take part in
Assumption	Discussion	Group	Order	Test for
Breakdown	Distinction	Highlight	Organize	Theme
Categorize	Distinguish	In-depth	Point out	
Cause & effect	Dissect	Inference	Research	
Choose	Divide	Inspect	See	
Classify	Establish	Isolate	Select	
Comparing	Examine	Investigate	Separate	

ii. Synthesis Skills – Synthesis skills involve changing or creating something new, compiling information together in a different way by combining elements in a new pattern proposing alternative solutions.

iii. Evaluation Skills – Evaluation skills involve justifying and presenting and defending opinions by making judgments about information, validity of ideas or quality of work based on set criteria.

Types of Values

Personal engagement and civic engagement strategies help young people to acquire and apply skills and dispositions that will prepare them to become competent and responsible citizens.

1. Personal Values (importance, worth, usefulness, etc.)

Core values	Sustaining values
• Sanctity of life • Truth • Aesthetics • Honesty • Human • Dignity • Rationality • Creativity • Courage • Liberty • Affectivity • Individuality	• Self-esteem • Self-reflection • Self-discipline • Self-cultivation • Principal morality • Self-determination • Openness • Independence • Simplicity • Integrity • Enterprise • Sensitivity • Modesty • Perseverance

2. Social Values

Core values	Sustaining values
• Equality • Kindness • Benevolence • Love • Freedom • Common good • Mutuality • Justice • Trust • Interdependence • Sustainability • Betterment of human kind • Empowerment	• Plurality • Due process of law • Democracy • Freedom and liberty • Common will • Patriotism • Tolerance • Gender equity and social inclusion • Equal opportunities • Culture and civilisation • Heritage • Human rights and responsibilities • Rationality • Sense of belonging • Solidarity • Peace and harmony • Safe and peaceful communities

Types of Attitudes

Attitudes - Ways of thinking and behaving, points of view	
• Optimistic • Participatory • Critical • Creative • Appreciative • Empathetic • Caring and concern • Positive • Confident • Cooperative	• Responsible • Adaptable to change • Open-minded • Diligent • With a desire to learn • With respect for self, life, equality and excellence, evidence, fair play, rule of law, different ways of life, beliefs and opinions, and the environment.

Teaching and Learning Strategies

Mathematics teaching emphasises and embraces the use of cognitive, reasoning, decision-making, problem solving and higher level thinking skills to teach to enhance students' understanding of inter-disciplinary concepts and issues in relation to environment, geography, history, politics and economic within PNG and globally. It aims to provide a meaningful pedagogical framework for teaching and learning essential and in demand knowledge, skills, values, and attitudes that are required for the preparation of students for careers, higher education and citizenship in the 21st Century.

Students must be prepared to gather and understand information, analyse issues critically, learn independently or collaboratively, organize and communicate information, draw and justify conclusions, create new knowledge, and act ethically.

These teaching and learning strategies will help teachers to;

- familiarize themselves with different methods of teaching in the classroom.
- develop an understanding of the role of a teacher for application of various methods in the classroom.

Successful teachers always keep in view that teaching must "be dynamic, challenging and in accordance with the learner's comprehension. He/she does not depend on any single method for making his/her teaching interesting, inspirational and effective.

A detailed table of Teaching and Learning Strategies are outlined below:

STRATEGY	TEACHER	STUDENTS
CASE STUDY Used to extend students' understanding of real life issues.	Provide students with case studies related to the topic of the lesson and allow them to analyse and evaluate.	Study the case study and identify the problem addressed. They analyse the problem and suggest solutions supported by conceptual justifications and make presentations. This enriches the students' existing knowledge of the topic.
DEBATE A method used to increase students' interest, involvement and participation.	Provide the topic or question of debate on current issues affecting a bigger population, clearly outlining the expectations of the debate. Explain the steps involved in debating and set a criteria/ standard to be achieved.	Conduct researches to gather supporting evidence about the selected topic and summarising the points.
DISCUSSION The purpose of discussion is to educate students about the process of group thinking and collective decision.	The teacher opens a discussion on certain topic by asking essential questions. During the discussion, the teacher reinforces and emphasises on important points from students responses.	Students ponder over the question and answer by providing ideas, experiences and examples.

STRATEGY	TEACHER	STUDENTS
	Teacher guide the direction to motivate students to explore the topic in greater depth and the topic in more detail. Use how and why follow-up questions to guide the discussion toward the objective of helping students understand the subject and summarise main ideas.	
GAMES AND SIMULATIONS Encourages motivation and creates a spirit of competition and challenge to enhance learning.	Being creative and select appropriate games for the topic of the lesson. Give clear instructions and guidelines. The game selected must be fun and build a competitive spirit to score more than their peers to win small prices.	Go into groups and organize. Follow the instructions and play to win.
OBSERVATION Method used to allow students to work independently to discover why and how things happen as the way they are. It builds curiosity.	Give instructions and monitor every activity students do	Students possess instinct of curiosity and are curious to see the things for themselves and particularly those things which exist around them. A thing observed and a fact discovered by the child for himself becomes a part of mental life of the child. It is certainly more valuable to him than the same fact or facts learnt from the teacher or a book. Students • Observe and ask essential questions • Record • Interpret.
PEER TEACHING & LEARNING (power point presentations, pair learning) Students teach each other using different ways to learn from each other. It encourages; team work, develops confidence, feel free to ask questions, improves communication skills and most importantly develop the spirit of inquiry.	Distribute topics to groups to research and teach others in the classroom. Go through the basics of how to present their peer teaching.	Go into their established working groups. Develop a plan for the topic. Each group member is allocated a task to work on. Research and collect information about the topic allocated to the group. Outline the important points from the research and present their findings in class.

STRATEGY	TEACHER	STUDENTS
PERFORMANCE-RELATED TASKS (dramatization, song/lyrics, wall magazines) Encourages creativity and take on the overarching ideas of the topic and are able to recall them at a later date	Students are given the opportunity to perform the using the main ideas of a topic. Provide the guidelines, expectations and the set criteria	Go into their established working groups. Being creative and create dramas, songs/lyrics or wall magazines in line with the topic.
PROJECT (individual/group) Helps students complete tasks individually or collectively	Teacher outline the steps and procedures of how to do and the criteria	Students are involved in investigations and finding solutions to problems to real life experiences. They carry out researches to analyse the causes and effects of problems to provide achievable solutions. Students carefully utilise the problem-solving approach to complete projects.
USE MEDIA & TECHNOLOGY to teach and generate engagement depending on the age of the students	Show a full movie, an animated one, a few episodes form documentaries, you tube movies and others depending on the lesson. Provide questions for students to answer before viewing	Viewing can provoke questions, debates, critical thinking, emotion and reaction. After viewing, students engage in critical thinking and debate

Strands, Units and Topics Suggested Lesson Titles

This section contains the overview of Mathematics content to be taught in Grade 12 Advanced Mathematics. The table below outlines strands, units, topics with suggested lesson titles. Teachers will use this to develop their own termly and yearly programs.

Strand	Unit	Topic	Suggested Lesson Titles
1. Number, Operation and Computation	Number and operations	Introduction to complex numbers	• Definition and Notations for complex numbers • The need for complex numbers and introduction of *i*. • Powers of *i* and properties of complex numbers • Conjugate and square roots of complex numbers
		Operations with Complex Numbers	• Addition and subtraction of complex numbers • Multiplication and Division of complex numbers
		The Argand Diagram	• Representing a Complex number on the Argand diagram • Calculating its Modulus and Argument
		Conversion of complex numbers in different forms	• Standard form to Polar coordinates • Polar coordinates to standard form
		Operations with complex numbers and De Moivre's Theorem	• General Formulae for Multiplication and Division of complex numbers in Polar form • Power form of complex numbers
		Quadratic Equations and Complex Numbers	• Solve quadratic equations using complex numbers • Solving other polynomial equations using complex numbers • Quadratic equations with real and complex roots • Solve problems with complex numbers
2. Geometry, Measurement and Transformation	Trigonometry	Radians and Degrees	• Identify types of angles • Degrees and radian measures of angles • Conversion of radians to degrees • Conversion of degrees to radians • Calculation of arc length • Calculation of area to a sector
		Relationships and sketching of trigonometric functions	• Relationship between $y = sin\, x$ and $y = \sin^{-1} x$ • Relationship between $y = \cos x$ and $y = \cos^{-1} x$ • Relationship between $y = \tan x$ and $y = \tan^{-1} x$ • Sketch trigonometric functions $y = \sin x,\quad y = \cos x,\quad y = \tan x$
		Trigonometric Equations	• Introduction to trigonometric equations • Values of trigonometric functions • Values of inverse trigonometric functions • Solve trigonometric equations using algebraic method
		Trigonometric Identities and Equations	• Introduction to fundamental trigonometric identities • Solve trigonometric equations involving identities
		Sum and Differences Formula	• Sum and Differences of Identities in solving trigonometric problems • Use Double-Angle Identities in solving problems
		Verification of trigonometric Identities	• Use Half-Angle Identities in solving trigonometric problems • Solve trigonometric problems • Proof equations involving Ratio Identities

	Trigonometry		• Proof equations involving Reciprocal Identities • Proof equations involving Pythagoras' Theorem • Proof equations involving Double-Angle Identities • Proof equations involving Half-Angle Identities
		Trigonometric Equations with and without restricted domains	• Trigonometric equations with all possible solutions • Trigonometric equations with restricted domains • Inverse trigonometric equations with all possible solutions • Inverse trigonometric equations with restricted domains
		Graphs of Sine and Cosine Functions	• Identify the domain and range of the sine, cosine and tangent functions • Introduction to the amplitude and period fo the sine and cosine functions • The phase shift (vertical translation) of the sine and cosine functions • The phase shift (horizontal translation) of the sine and cosine functions • The amplitude, period, phase shift and horizontal shift of the tangent functions • Using the graphs of $y = A\sin(Bx + C) + D$ and $y = A\cos(Bx + C) + D$ to determine the values of the variables.
		Authentic Problems with sine functions	• Real life problems involving the sine function $y = A\sin(Bx + C) + D$
3. Patterns and Algebra	Sequence and Series	The nth term and sum of n terms of the Arithmetic Sequence	• The nth term of the Arithmetic Sequence • Sum of n terms of an Arithmetic Sequence • Use Sigma Notation
		The *nth* term and sum of *n* terms of the Geometric Sequence	• The *nth* term of the Geometric Sequence • Sum of *n* terms of a Geometric Sequence • Use Sigma Notation
		Properties of Arithmetic and Geometric Sequences and Series	• Solve problems using the properties of Arithmetic Sequences • Solve problems using the properties of Geometric Sequence
		Infinite Geometric Series	• Limiting sum of geometric series
		Pascal's Triangle	• Use Pascal's triangle to expand binomial expression
		Factorial, Permutation and Combination	• Factorial and Combination • Permutation
	Binomial Theorem	Binomial Theorem	• Introduction to Binomial Theorem and its properties • The coefficient of the nth term in the binomial expression • The nth term of the binomial expression
	Determinants	Determinants	• Introduction to Determinants • Determinants of a two by two system • Determinants by three by three system • Solution to systems of linear equation using determinants
	Limits	Finding limits numerically and graphically	• Definition of limits and its numerical representation • Graphical representation of limits
		Finding limits algebraically using the limit laws	• Properties of limits • Addition and Constant Theorems • Multiplication and Division Theorems • Limits of Rational Functions

		Continuity and Limits	• Definition of continuity and its properties • Continuous functions • Limits involving infinity
	Differentiation	The Gradient Function	• Introduction to Differentiation • The limiting case of the secant line and the tangent to a curve
		Definition of Derivatives from First Principle	• Secant and Tangent line and Derivative at a point • Using the definition of Derivatives • Gradient and Equation of tangent lines
		Standard Rules of Differentiation	• Standard Rules of Differentiation • Use of the Standard Rules of Differentiation
		General Rules of Differentiation	• Product and Quotient Rules • Function of a function Rules • Derivative of Rational Functions • The Chain Rule for composite functions
		Derivatives of Exponential and Logarithmic functions	• Derivatives of Exponential functions • Derivatives of Logarithmic functions
		Derivatives of the trigonometric functions	• Derivatives of trigonometric functions and their reciprocals
		Implicit Differentiation	• Finding derivatives of relations and functions using implicit differentiation method
		Maxima, Minima, Tangents, normal and Stationery point	• Normal and tangents • Stationery points, maximum and minimum turning points
		The second derivatives	• Second derivatives test for maximum or minimum and increasing or decreasing • Curve Sketching
		Application of Calculus	• Optimisation – Application of maximum and minimum ideas
	Integration	Definition of Area under the curve	• Introduction to integration • Constant, sum and differences theorems and constant of integration
		Rules of Integration	• The Power Rule
		Anti-derivatives and indefinite integrals	• Finding the anti-derivatives • Indefinite integrals
		Integrals of Trigonometric and exponential functions	• Integrals of derivatives of standard trigonometric functions, $\sin\theta$, $\cos\theta$ and $\tan\theta$ • Integrals of exponential functions • Evaluating definite integrals
		Application - Area of Curves	• Area bounded by a line and the axes • Area bounded by a line and a curve • Area bounded by two curves
4. Statistics and Probability	Bivariate Statistics	Bivariate data on scatter diagrams	• Scatter Diagrams • Linear and non-linear Relationship
		Linear Regression	• Residuals • Equation of Regression Line
		Correlation	• Types of Correlation • Regression and Correlation Coefficient
		Use of technology in problem solving	• Using a Calculator • Using an Excel Function

Strand 1: Number, Operations and Computation

Content Standard: Students will be able to represent numbers in various situation and forms, develop fluency in calculations through operations, use base ten as key for extending numbers and operations, and apply numbers in practical situations to develop number sense

Units	Benchmark	Topics	Lesson Titles
Number and Operation	**12.1.1.1** Use the complex number system, the notation for complex numbers, and the definition of *i* to solve problems in standard form.	Introduction to Complex numbers	Definition and Notation for Complex Numbers. The Need for Complex Numbers and Introduction of *i*.
			Powers of *i* and Properties of Complex Numbers.
			Conjugates and Square Roots of Complex Numbers
	12.1.1.2 Add, subtract, multiply, and divide complex numbers and represent a complex number on a Argand Diagram to calculate its Modulus and Argument.	Operations with Complex Numbers	Addition and Subtraction of Complex Numbers.
			Multiplication and Division of Complex Numbers.
		The Argand Diagram	Representing a Complex Number on the Argand Diagram.
			Calculating its Modulus and Argument.
	12.1.1.3 Express complex numbers in standard and polar form and convert one to another.	Conversion of complex numbers in different forms	Standard form to Polar Coordinates.
			Polar Coordinates to Standard forms.
	12.1.1.4 Multiply and divide complex numbers in polar form and find powers of complex numbers.	Operations with Complex numbers and De Moivre's Theorem	General Formulas for Multiplication and Division of Complex Numbers in Polar Form.
			Power Form of Complex Numbers.
	12.1.1.5 Apply quadratic equations to solve real-world situations and complex number problems.	Quadratic Equations and Complex Numbers	Solve quadratic equations using complex numbers.
			Solving other Polynomial Equations using complex numbers.
			Quadratic Equations with Real and complex roots.
			Solve problems with complex numbers.

Strand 1: Number, Operations and Computation

Unit: Number and Operation | **Topic : Introduction to Complex Numbers**

Content Standard:	Students will be able to represent numbers in various situation and forms, develop fluency in calculations through operations, use base ten as key for extending numbers and operations, and apply numbers in practical situations to develop numbers sense.
Benchmark	**12.1.1.1** Use the complex number system, the notation for complex numbers, and the definition of "*i*" to solve problems in standard form.

Learning Objective: By the end of the topic, students will be able to;



- define and give a brief account of the history to complex numbers,
- identify the significance of complex numbers and its notations, and
- explain the value of '*i*' and how it is used.

Essential questions:



- What are complex numbers?
- How was the concept of complex numbers developed?
- How are complex numbers significant?

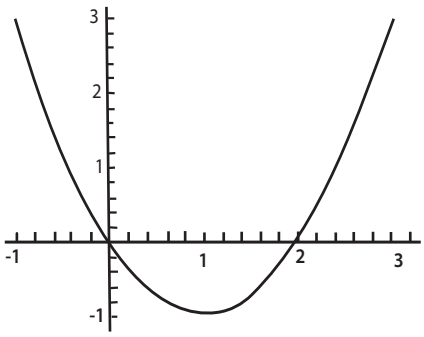
Key Concepts(ASK-MT)

Attitudes/Values	Acknowledging that there is complex number existing which is useful in some mathematical computation making math fun and exciting.
Skills	Representing complex numbers by using the appropriate notations.
Knowledge	i. Origination of Complex Numbers and its purpose in the number system. ii. Know that real numbers are part of Complex Numbers, the notations, the imaginary (i), conjugate and square roots of complex numbers.
Mathematical Thinking	Think about how "*i*" was created to solve problems such as "$\sqrt{-39}$" and "$x^2+1=0$".

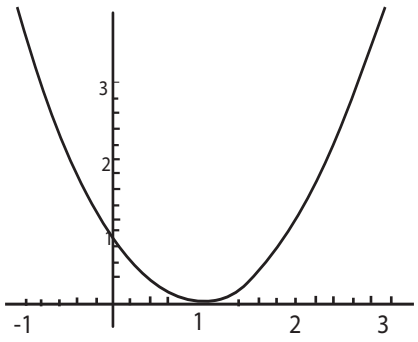
Content Background

1. The Need for Complex Numbers

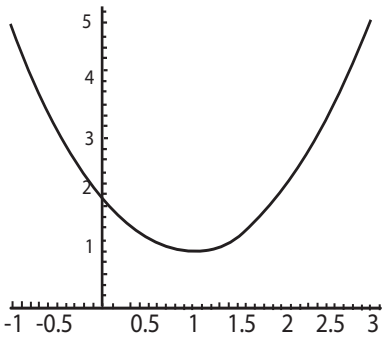
We all know that the two roots of the equation $ax^2+bx+c=0$ are $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ and solving quadratic equations is something that mathematicians have been able to do since the time of the Babylonians. When $b^2-4ac>0$ then these two roots are real and distinct; graphically they are where the curve $y=ax^2+bx+c$ cuts the x-axis. When $b^2-4ac=0$ then we have one real root (identical, coincidental, repeated or equal roots) and the curve just touches the x-axis. But what happens when $b^2-4ac<0$? In this case there are no real solutions (roots) to the equation, as no real number squared gives the negative b^2-4ac. From the graphical point of view, the curve $y=ax^2+bx+c$ lies entirely above or below the x-axis.



Two Distinct Real Roots



One Repeated Real Root



No Real Roots

It is only comparatively recently that mathematicians have been comfortable with these roots when $b^2 - 4ac < 0$. During the Renaissance the quadratic would have been considered unsolvable, or its roots would have been called imaginary.

If we imagine $\sqrt{-1}$ to exist, and that it behaves much like other numbers, then the two roots of the quadratic $ax^2 + bx + c = 0$ can be written in the form $x = A + B\sqrt{-}$, where $A = -\frac{b}{2a}$ and $B = \frac{\sqrt{4ac - b}}{2a}$ are real numbers.

But what meaning can such roots have? It was this philosophical point which pre-occupied mathematicians until the start of the 19th century; afterwards these 'imaginary' numbers started proving so useful.

Notation: We shall from now on write *i* for $\sqrt{-1}$ – this is standard notation amongst mathematicians, though many books, particularly those written for engineers and physicists, use j instead.

Definition: A complex number is a number of the form $a + bi$ where a and b are real numbers. If $z = a + bi$ then a is known as the real part of z and b as the imaginary part. We write a = Re z and b = Im z. Note that real numbers are complex – a real number is simply a complex number with zero imaginary part.

Using the quadratic formula,` any quadratic equations can be solved using complex numbers.

Examples

1. Solve if possible, the following quadratic equations by factorizing or by using the quadratic formula. If a solution is not possible explain why.

(a) $x^2 - 1 = 0$ (b). $x^2 - x - 6 = 0$ (c). $x^2 - 2x - 2 = 0$ (d). $x^2 - 2x + 2 = 0$

Solution

You should have found (a), (b) and (c) straight forward but in (d) a term appears in the solution which includes the square root of a negative number and to obtain the solutions you need to use the symbol $\mathrm{i} = \sqrt{-1}$ or $i^2 = -1$

It is then possible to obtain solution to (d) in question 3 below.

2. Find $\sqrt{-25}$

Solution

$\sqrt{-25}$ = 5*i*

since

$$(5i)^2 = 5^2 \mathrm{x}\, i^2$$
$$= 25\, x\, (-1)$$
$$= -25$$

3. Solve $x^2 - 2x + 2 = 0$

Solution: Using the quadratic formula

$$x = \frac{b \pm \sqrt{b^2 \quad 4ac}}{2a}$$

$$\Rightarrow x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} \Rightarrow x = \frac{2 \pm \sqrt{-4}}{2a}$$

But $\sqrt{-4} = \sqrt{4(-1)} = \sqrt{4}\sqrt{-1} = 2\sqrt{-1} = 2i$

$$\therefore \quad x = \frac{2 + 2i}{2} \Rightarrow x = 1 \pm i$$

Therefore the two solutions are $x = 1 + i$ and $x = 1 - i$

Properties of Complex Number System

1. The Complex number system contains all real numbers.
2. The complex number system contains i such that i^2 = -1.
3. Addition, subtraction, multiplication, and division of complex numbers obey the same rules of arithmetic that hold in real number system, with one exception: The exponent laws hold for integer exponents, but not necessarily for fractional ones.
4. Every complex number can be written in standard form a + b i , where a and b are real numbers.
5. Two complex numbers a + b i and c + d i are equal exactly when a = c and b = d.

Unit: Number and Operation	Topic : Operations with Complex Numbers
Benchmark:	**12.1.1.2** Add, subtract, multiply, and divide complex numbers and represent a complex number on the Argand Diagram to calculate its Modulus and Argument

Learning Objective: By the end of the topic, students will be able to;

- add and subtract complex numbers,
- multiply and divide complex numbers.



Essential questions:

- Can complex numbers be added,subtracted, multiplied or divided as real numbers?
- How are complex numbers added,subtracted, multiplied or divided?



Key Concepts(ASK-MT)	
Attitudes/Values	Appreciating that complex numbers can also be evaluated through addition, subtraction, multiplication and division.
Skills	Computing of complex numbers through addition, subtraction, multiplication and division.
Knowledge	Addition, subtraction, multiplication and division of complex numbers.
Mathematical Thinking	Think about how to add, subtract, multiply and divide complex numbers.

Content Background

Addition and Subtraction of Complex Numbers

When adding complex numbers, add the real parts and add the imaginary parts together

Imaginary Part

$(3 + 7i) + (8 + 11i)$

Real Part

$11 + 18i$

When subtracting complex numbers, be sure to distribute the subtraction sign; then add like parts.

$(5 + 10i) - (15 - 2i)$

$5 + 10i - 15 + 2i$

$-10 + 12i$

Example: Write the expression as a complex number in standard form.

a. $(4-i)+(3-2i)$ **b.** $(7-5i)-(1-5i)$ **c.** $6-(-2+9i)+(-8+4i)$

Solution

a. $(4-i)+(3-2i) = (4+3)+(-1-2)$

$= 7-3i$

Definition of complex addition
Standard Form

b. $(7-5i)-(1-5i) = (7-1)-(-5+)i$

$= 6+0i$

$= 6$

Definition of complex subtraction
Simplify
Standard form

c. $6-(-2+9i)+(-8+4i) = [(6+2)-9i]+(-8+4i)$

$= (8-9i)+(-8+4i)$

$= (8-8)+(-9+4)i$

$= 0-5i$

$= -5i$

Subtract
Add
Simplify
Standard form

Multiplication of Complex Numbers

We multiply two complex numbers just as we would multiply expressions of the form $(x+y)$ together.

$$\begin{aligned}(a+ib)(c+id) &= ac + a(id) + (ib)c + (ib)(id)\\ &= ac + iad + ibc - bd\\ &= ac - bd + i(ad+bc)\end{aligned}$$

Example:

$(2+3i)(3+2i)$

$(2+3i)(3+2i) = 2(3) + 2(2i) + 3i(3) + 3i(2i) = 6 + 4i + 9i + 6i^2 = 6 + 13i - 6 = 13i - 6 = 13i$

Division of Complex Numbers

The trick for dividing two complex numbers is to multiply top and bottom by the complex conjugate of the denominator. The product of a complex number and its conjugate results in a real number.

Example:

(i) $\dfrac{1}{i} = \dfrac{1}{i} \text{X} \dfrac{i}{i}$

$= \dfrac{i}{i \text{ x } i}$

$= \dfrac{i}{-1}$

$= -i$

(ii) $\dfrac{(2+3i)}{(1+2i)} = \dfrac{(2+3i)(1-2i)}{(1+2i)(1-2i)}$

$= \dfrac{(2+3i)(1-2i)}{1+4}$

$= \dfrac{1}{5}(2+3i)(1-2i)$

$= \dfrac{1}{5}(2-4i+3i+6)$

$= \dfrac{1}{5}(8-i)$

Unit: Number and Operation | **Topic : The Argand Diagram**

Benchmark: **12.1.1.2** Add, subtract, multiply, and divide complex numbers and represent a complex number on the Argand Diagram to calculate its Modulus and Argument.

Learning Objective: By the end of the topic, students will be able to;

- represent and explain a complex number on the Argand Diagram.



Essential questions:

- What is a Argand Diagram?
- How can complex numbers be represented on a Argand Diagram?



Key Concepts(ASK-MT)	
Attitudes/Values	Appreciate how to solve expressions containing imaginary values and explain what it means.
Skills	Computing complex numbers and representing on a Argand Diagram
Knowledge	Complex Numbers and Argand Diagram
Mathematical Thinking	Think about parts of complex numbers, the notations, the imaginary (*i*), conjugate and square roots of complex numbers.

Content Background

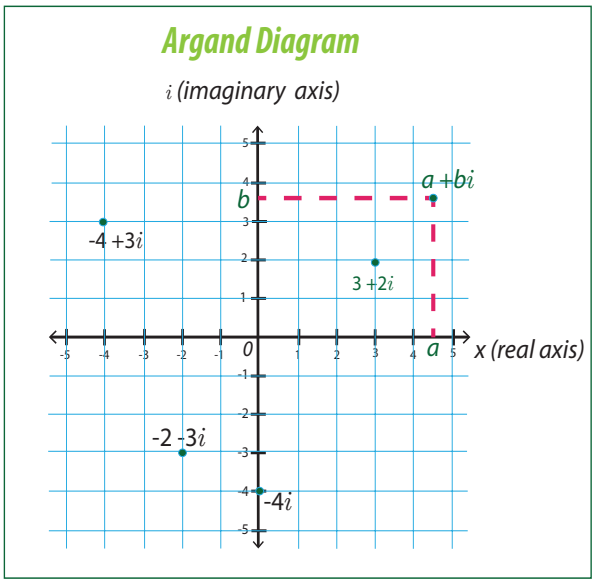
What is an Argand Diagram?

The Argand Diagram was devised by the Swiss mathematician Jean Robert Argand in 1806. A similar representation had been proposed by the Danish surveyor Caspar Wessel in 1797, but this was not generally known until later.

An Argand Diagram is defined as a plot of complex numbers as points. It has two axes, like a regular Cartesian graph. However, in this case the horizontal axis represents real number and the vertical axis represents imaginary numbers. The axes cross at zero, again just like in a Cartesian graph.

To plot $3+2i$ on an Argand diagram, you plot the point where the value on the real axis reads 3 and the value on the imaginary axis reads $2i$. Then, extend a line from 0 to the point you just plotted. That line is the visual representation of the number $3+2i$.

Example: Calculate the moduli (modulus – singular) and arguments of the complex numbers $z_1 = 3+4i$ *and* $z_2 = -4+2i$ and show this in the Argand diagram.



Solution: The modulus is simply the length of the line segment that goes from the origin to the terminal point of the complex number. If $z = a+bi = x+yi$ then the modulus is denoted as $|z|$ and is found by $|z| = \sqrt{a^2+b^2} = \sqrt{x^2+y^2}$ The argument is the angle the line segment (similar to vector) makes with the positive x – axis. In the case of the complex number $z_1 = 3+4i$ below, its argument is α . The argument of the complex number $z_2 = -4+2i$ is $180^0 - \beta$. If you look at the diagram below, the line segment representing the complex number will make a perfect right triangle with the x – axis. The modulus is simply

the hypotenuse of the right triangle. The argument can be found by application of trigonometry. Let us proceed with finding the modulus and argument of the two given complex numbers.

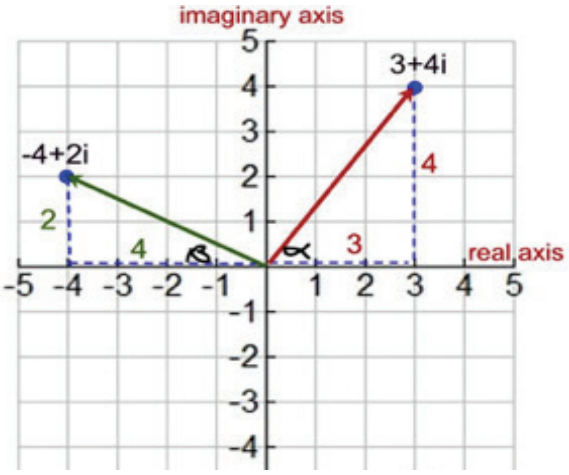
$$\textit{Modulus of } z_1 = |z_1| = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$\textit{Argument}: \tan\alpha = \frac{4}{3}. \alpha = 53^0 \,(\textit{nearest})$$

$$\textit{Modulus of } z_2 = |z_2| = \sqrt{(-4)^2 + 2^2} = \sqrt{16} + 4 = \sqrt{20} = 2\sqrt{5}$$

$$\textit{Argument}: \tan\beta = \frac{2}{4}. \beta = 27^0 \,(\textit{nearest})$$

$$\textit{Argument of } z_2 = 180^0 - 27^0 = 153^0.$$



Unit: Number and Operation	Topic : Conversion of complex numbers in different forms
Benchmark	**12.1.1.3** Express complex numbers in standard and polar form and convert one to another.

Learning Objective: By the end of the topic, students will be able to;

- convert complex numbers in standard form to Polar form and vice versa.



Essential questions:

- What is a standard form?
- What is a polar form?
- How do you convert from standard form to polar form?

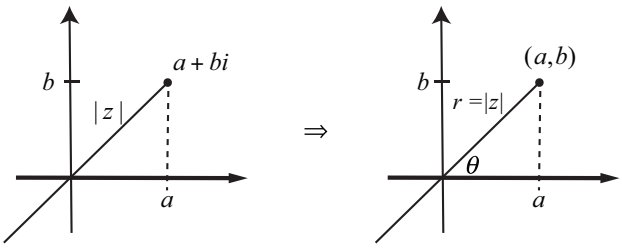


Key Concepts(ASK-MT)	
Attitudes/Values	Clearly understanding the relationship between the standard form of complex number and polar form which have the same meaning.
Skills	Conversion of the different forms of expressing complex numbers.
Knowledge	Conversion of Complex numbers in either standard or polar forms.
Mathematical Thinking	Think about how to convert complex numbers either from polar to standard form or from standard to polar form in any given situation.

Content Background

Complex numbers are numbers in which the real component and the imaginary part of the number are both represented. The numbers in standard form will be $a+bi$, where a is the real part and bi is the imaginary part. An example of a complex number would be $3+5i$. 3 is the real part, and $5i$ is the imaginary part.

In complex number representation, the standard form $a+bi$ is also known as the rectangular form. A complex number in the form of $a+bi$, whose point is (a,b), is in rectangular form and can therefore be converted into polar form just as we need with the points (x,y). The relationship between a complex number in rectangular form and polar form can be made by letting θ to be the angle (in standard position) whose terminal side is the point (a,b).



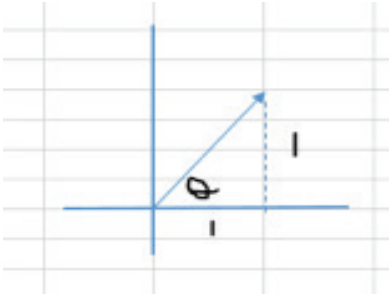
$\sin\theta = \frac{b}{r}$ $\quad r\sin\theta = b$ $\quad \cos\theta = \frac{a}{r}$ $\quad r\cos\theta = a$ $\quad \tan\theta = \frac{b}{a}$

$r = |z| = \sqrt{a^2 + b^2}$

Using these relationships, we can convert the complex number z form its rectangular form to its polar form.

$$z = a + bi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = rcis\theta$$

For instance express in polar form. We need to find two things; (i) the modulus and (ii) the argument and just substitute into the above formula. The position of the terminal point (1,1) is in the first quadrant.



$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$Arg(z): \tan\theta \frac{1}{1} : \theta = 45^0 = \frac{\pi}{4}$$

$$z = \sqrt{2}\frac{\pi}{4} + i\sqrt{2}\sin\frac{\pi}{4} = \sqrt{2}\,cis\frac{\pi}{4}$$

Converting from Polar Form to Rectangular Form

Either method of notation is valid for complex numbers. The primary reason for having two methods of notation is for ease of longhand calculation, rectangular form lending itself to addition and subtraction, and polar form lending itself to multiplication and division. Conversion between the two notational forms involves simple trigonometry. To convert from polar to rectangular, find the real component by multiplying the polar magnitude by the cosine of the angle, and the imaginary component by multiplying the polar magnitude by the sine of the angle. This may be understood more readily by drawing the quantities as sides of a right triangle, the hypotenuse of the triangle representing the vector itself (its length and angle with respect to the horizontal constituting the polar form), the horizontal and vertical sides representing the "real" and "imaginary" rectangular components, respectively.

For instance, express the complex number $z = \sqrt{3}\cos 120^0 + i\sqrt{3}\sin 120^0$ in rectangular form.

A complex number in rectangular form is $z = x + yi$. We have the following formula $x = r\cos\theta$ and $y = r\sin\theta$ where $r = \sqrt{x^2 + y^2}$. The argument is 120⁰ which means the terminal point is in the second quadrant in which the real part is negative and the complex part is positive. This is confirmed from trigonometry in which cosine is negative in the second quadrant while sine is positive. $r = \sqrt{3}$ and we just need to find $\cos 120^0$ and $\sin 120^0$. The reference angle in the first quadrant is 60⁰ . $\cos 60^0 = \frac{1}{2} and \sin 60^0 = \frac{\sqrt{3}}{2}$. Since real part is negative, we have $x = \sqrt{3}\,\mathrm{x}\left(-\frac{1}{2}\right) = -\frac{\sqrt{3}}{2}$.

$y = \sqrt{3}\,\mathrm{x}\ \frac{\sqrt{3}}{2} = \frac{3}{2}$. Hence the rectangular form of the given complex number is $z = -\frac{\sqrt{3}}{2} + \frac{3}{2}i.$

Unit: Number and Operation	Topic: Operations with Complex numbers and De Moivre's Theorem
Benchmark	**12.1.1.4** Multiply and divide complex numbers in polar form and find powers of complex numbers.

Learning Objective: By the end of the topic, students will be able to;

- multiply and divide complex numbers in polar forms.



Essential questions:

- How are complex numbers multiplied or divided while in polar forms?
- What are the general formulae for multiplication and division for complex numbers?
- What is De Moivre's Theorem?



Key Concepts(ASK-MT)	
Attitudes/Values	Appreciate that complex numbers expressed in polar form can be multiplied and divided through the formulae and also apply the De Moivre's Theorem.
Skills	Multiplying and dividing complex numbers.
Knowledge	De Moivre's Theorem, multiplication and division of complex numbers.
Mathematical Thinking	Think about how to calculate complex numbers through multiplication and division and also the application of De Moivre's Theorem and powers of complex numbers.

Content Background

1. Definition of Polar Form of a Complex Number

The polar form of nonzero complex number $z = a + bi$ is given by $z = r(\cos\theta + \sin\theta)$, where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = \frac{b}{a}$. The number r is the modulus of z and θ is the argument of z.

2. Product and Quotient of Two Complex Numbers

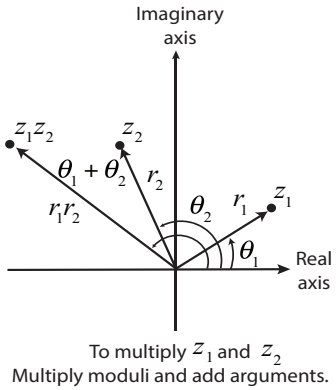
Given two complex numbers in a polar form

$z_1 = r_1\cos\theta_1 + ir\sin\theta_1$ and $z_2 = r_2\cos\theta_2 + ir_2\sin\theta_2$, the product and quotient of the numbers are as follows.

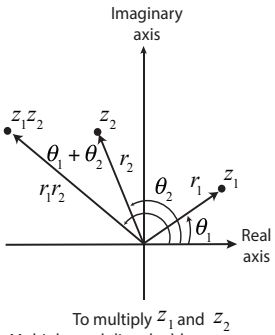
$$z_1 z_2 = r_1 r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$ Product

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right],\ z_2 \neq 0$$ Quotient

This theorem says that to multiply two complex numbers in polar form, multiply moduli and add arguments, and to divide two complex numbers, divide moduli and subtract arguments.
(See diagram representations on right)



To multiply z_1 and z_2
Multiply moduli and add arguments.



To multiply z_1 and z_2
Multiply moduli and add arguments.

Example: Determine $z_1 z_2$ and $\frac{z_1}{z_2}$ for the complex numbers

$$z_1 = 5\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right). \text{ and } z_2 = \frac{1}{3}\left(\cos\frac{\pi}{6} + \sin\frac{\pi}{6}\right)$$

Solution : Because you are given the polar forms of z_1 and z_2 you can apply the product and quotient of complex numbers as follows.

$$z_1 z_2 = (5)\left(\frac{1}{3}\right)\left[\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right]\right. = \frac{5}{3}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

$$\frac{z_1}{z_2} = \frac{5}{1/3}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right] = 15\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

Multiplying and Dividing in Polar Form

DeMoivre's Theorem:

Repeated use of multiplication in the polar form yields.

$$z = r(\cos\theta + i\sin\theta)$$

$$z^2 = r(\cos\theta + i\sin\theta)r(\cos\theta + \sin\theta) = r^2(\cos 2\theta + i\sin 2\theta)$$

$$z^3 = r(\cos\theta + i\sin\theta)r^2(\cos 2\theta + i\sin 2\theta) = r^3(\cos 3\theta + i\sin 3\theta).$$

Similarly,

$$z^4 = r^4(\cos 4\theta + i\sin 4\theta)$$

$$z^5 = r^5(\cos 5\theta + i\sin 5\theta)$$

Raising a Complex Number to an Integer Power

DeMoivre's Theorem

If $z = r(\cos\theta + \sin\theta)$ and n is any positive integer, then $Z^n = r^n(\cos n\theta + i\sin n\theta)$

Example: Find $(-1 + \sqrt{3i})^{12}$ and write the result in standard form.

Solution: First convert to polar form. For $-1 + \sqrt{3i},$

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2 \; and \tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

Which implies that =2π/3. So,

$$-1 + \sqrt{3i} = 2\left(\cos\frac{2}{3} + i\sin\frac{2\pi}{3}\right)$$

By DeMoivre's Theorem,

$$(-1 + \sqrt{3i})^{12} = \left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^{12}$$

$$= 2^{12}\left[\cos\frac{12(2\pi)}{3} + i\sin\frac{12(2\pi)}{3}\right]$$

$$= 409(\cos 8\pi + i\sin 8\pi)$$

$$= 4096[1 + i(0)] = 4096$$

Unit: Number and Operation	Topic: Quadratic Equations and Complex Numbers
Benchmark	**12.1.1.5** Apply quadratic equations to solve real-world situations and complex number problems.

Learning Objective: By the end of the topic, students will be able to;

- solve quadratic equations using complex numbers.



Essential questions:

- What is one way you can identify a quadratic equation?
- How are quadratic equations in complex numbers solved?
- Can quadratic equations using complex numbers be applicable in real life?



Key Concepts(ASK-MT)	
Attitudes/Values	Confidently using background knowledge or information in quadratic equations and incorporating it with complex numbers.
Skills	Solving complex numbers in the form of quadratic equations.
Knowledge	Knowing that complex numbers expressed in quadratic equations can be solved.
Mathematical Thinking	Obtaining solutions to Polynomial equations with complex numbers and complex roots.

Content Background

Quadratic equations with a negative discriminant have no real number solution. However, if we extend our number system to allow complex numbers, quadratic equations will always have a solution. Since the solution to a quadratic equation involves the square root of the discriminant, we begin with a discussion of square roots of negative numbers.

If N is a positive real number, we define the **principal square root of –N**, denoted by $\sqrt{-N}$ as $\sqrt{-N} = \sqrt{N}\,i$ where i is the imaginary unit and $i^2 = -1$

Theorem: Quadratic Formula

In the complex number system, the solutions of the quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Solving a Quadratic Equation in the Complex Number System

Solve the equation $x^2 - 4x + 8 = 0$ in the complex number system.

Solution: Here $a = 1, b = -4, c = 8, and\ b^2 - 4ac = 16 - 4(1)(8) = -16.$

Using the quadratic equation;

$$x = \frac{-(-4) \pm \sqrt{-16}}{2(1)} = \frac{4 \pm \sqrt{16i}}{2} = \frac{4 + 4i}{2} = 2 \pm 2i$$

The equation has two solutions $2-2i$ and $2+2i$. The solution set is $\{2-2i, 2+2i\}$.

Check: $2+2i: (2+2i)^2 - 4(2+2i) + 8 = 4 + 8i + 4i^2 - 8 - 8i + 8$

$= 4 - 4 = 0$

$2-2i: (2-2)^2 - 4(2-2i) + 8 = 4 - 8i + 4i^2 - 8 + 8i + 8$

$= 4 - 4 = 0$

Strand 2: Geometry, Measurement and Transformation

Content Standard: Students will be able to comprehend the meaning and significance of geometry, measurements and spatial relationship including units and system of measurement and develop and use techniques, tools, and formulas for measuring the properties of objects and relationships among the properties and use transformations and symmetry to analyse mathematical situations.

Units	Benchmark	Topics	Lesson Titles
Trigonometry	**12.2.2.1.** Translate between radians and degrees and calculate the arc length and area of sector.	Radians and Degrees	Identify types of angles.
			Degrees and Radian Measure of Angles
			Conversion of radians to degrees
			Conversion of degrees to radians
			Calculation of arc length
			Calculation of area to a sector
	12.2.2.2 Explain the relationship between trigonometric functions and their inverse and sketch the trigonometric functions	Relationships and Sketching of Trigonometric Functions	Relationship between $y=\sin x$ and $y=\sin^{-1}x$
			Relationship between $y=\cos x$ and $y = \cos^{-1}x$
			Relationship between $y = \tan x$ and $y = \tan^{-1}x$
			Sketch trigonometric functions ($y=\sin x$, $y=\cos x$)
			Sketch trigonometric function $y=\tan x$
	12.2.2.3 Find the value of a trigonometric function and inverse trigonometric function and solve trigonometric equations.	Trigonometric equations	Introduction to trigonometric equations
			Values of trigonometric functions
			Values of the inverse trigonometric functions
			Solve trigonometric equations using algebraic method
	12.2.2.4 Apply trigonometric Identities to find solutions to trigonometric equations.	Trigonometric Identities and Equations	Introduction to fundamental trigonometric identities
			Solve trigonometric equations involving identities
	12.2.2.5 Use fundamental trigonometric identities, including sum and difference formulas, double-angle formulas and half-angle formulas to solve problems.	Use of Trigonometric Identities	Sum and Differences of Identities in solving trigonometric problems
			Use Double-Angle Identities in solving trig problems
			Use Half-Angled Identities in solving trig problems
			Solve trigonometric problems
	12.2.2.6 Verify trigonometric identities.	Verification of trigonometric identities	Proof equations involving Ratio Identities
			Proof equations involving Reciprocal Identities
			Proof equations involving Pythagorean Identities
			Proof equations involving Double-Angled Identities
			Proof equations involving the Half-Angled Identities

	12.2.2.7 Solve trigonometric equations and inverse trigonometric equations that include all solutions or solutions with restricted domains.	Trigonometric Equations with and without restricted domains	Trigonometric equations with all possible solutions
			Trigonometric equations with restricted domains
			Inverse trigonometric equations with all possible solutions
			Inverse trigonometric equations will restricted domains
	12.2.2.8 Use the trigonometric functions in the form y =Asin (Bx+C)+D and y=A cos (Bx+C)+D to determine various properties of the function including domain, range, period, phase shift and amplitude.	Graphs of Sine and Cosine Function	Identify the domain and range of the sine, cosine and tangent functions
			Introduction to the amplitude and period of the sine and cosine functions
			The phase shift (vertical translations) of the sine and cosine functions
			The horizontal translation of the sine and cosine functions
			The amplitude, period, phase shift and horizontal shift of the tangent functions
			Using the graphs of y = Asin(Bx+C)+D and y = Acos(Bx+C)+D to determine the values of the variables.
	12.2.2.9 Identify real-world phenomena that can be represented by the trigonometric function in the form y=Asin (Bx+C)+D	Authentic Problems with sine functions	Real life problems involving the sine function y = Asin(Bx+C)+D

Unit: Trigonometry	Topic : Radians and Degrees
Benchmark	**12.2.2.1.** Translate between radians and degrees and calculate the arc length and area of sector.

Learning Objective: By the end of the topic, students will be able to;

- convert between radian and degree measures, and calculate arc length and area of sector.

Essential questions:

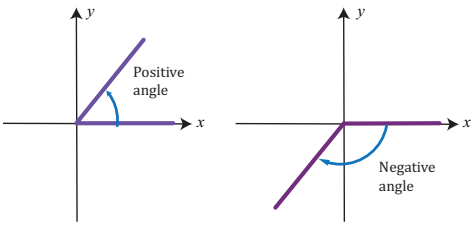
- What are arc lengths?
- Why should we know how to calculate arc length?
- How are arc lengths calculated?

Key Concepts(ASK-MT)	
Attitudes/Values	Experiencing the relationship between radians and degrees such that a value expressed in radians has its equivalent in degrees and vice versa.
Skills	Converting measures in radians to degrees and vice versa and also calculation of arc lengths and area of sectors.
Knowledge	Conversion of radian and degree measures, arc length and area of sector.
Mathematical Thinking	Translating between degrees and radian measures and obtaining arc length and area of sectors.

Content Background

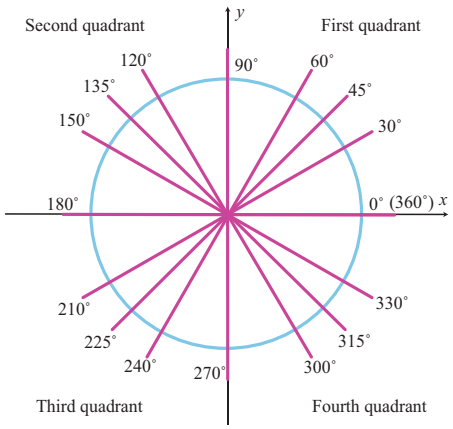
1. Angles in a circular function

Each angle is given in both degrees and radians. The radian measure of each angle is the same as the distance from (1, 0) to the point on the terminal side of the angle, as measured along the circumference of the circle in a counter-clockwise direction.



The x- and y-coordinate of each point shown are the cosine and sine, respectively, of the associated angle or distance.

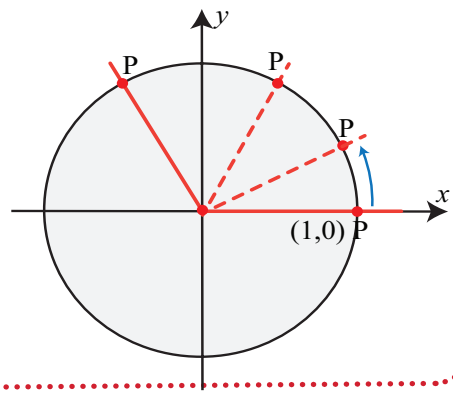
An angle in the coordinate plane is said to be in standard position if its vertex is at the origin and its initial side on the positive x – axis, as shown in the diagram on the right. When measuring angles in standard position, we use positive numbers for the angles obtained by counter-clockwise rotation (positive angles) and negative numbers for ones obtained by clockwise rotation (negative angles).



The classical unit for angle measurement is the degree (in symbol, °). You should be familiar with the positive angles in standard position shown below. Note that a 360° angle corresponds to one full revolution and thus is conterminal with an angle of 0°.

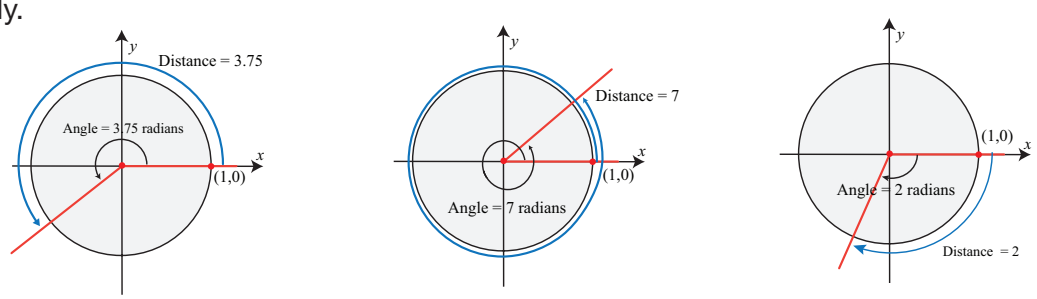
2. Radian Measure

Because it simplifies many formulas in calculus and physics, a different unit of angle measurement is used in mathematical and scientific applications. Recall that the unit circle of radius 1 with centre at the origin; its equation is $x^2 + y^2 = 1$. When a positive angle in standard position is formed by rotating the initial side (the positive x – axis) counter clockwise, then the point P = (1, 0) moves along the unit circle as shown in the figure on the right.



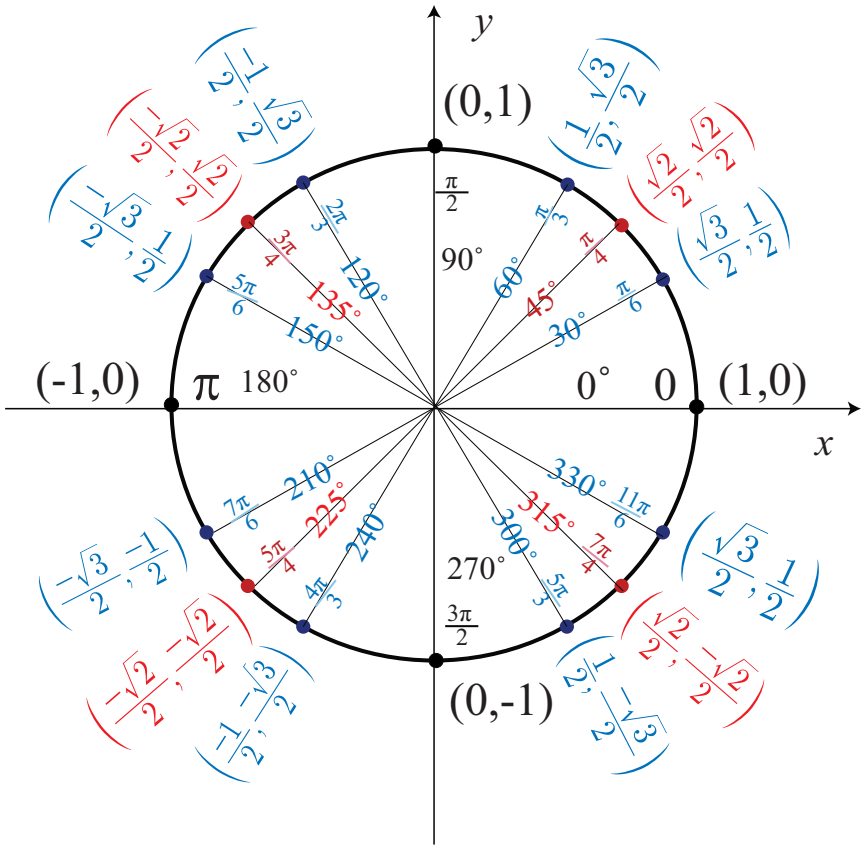
The ***radian measure*** of the angle is defined to be: ***the distance travelled along the unit circle by the point P as it moves from its starting position on the initial side to its final position on the terminal side of the angle.***

The radian measure of a negative angle in standard position is found in the same way, except that you move clockwise along the unit circle. The figure below shows angles 3.75, 7, and -2 radians respectively.

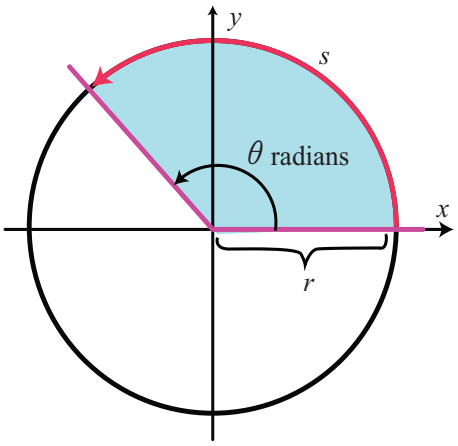


3. The Unit Circle $x^2 + y^2 = 1$

A unit circle is defined as a circle with a radius of 1. Often, especially in applications to trigonometry, the unit circle is centered at the origin (0,0) in the coordinate plane. The equation of the unit circle in the coordinate plane is $x^2 + y^2 = 1$. As mentioned above, the unit circle is taken to be 360°, or 2π radians. We can divide the coordinate plane, and therefore, the unit circle, into 4 quadrants. The first quadrant is defined in terms of coordinates by $x > 0$, $y > 0$, or, in terms of angles, by 0°< θ <90°, or 0< θ <π/2. The second quadrant is defined by x<0, y>0, or 90°< θ <180°, or π/2< θ <π. The third quadrant is defined by x<0, y<0, or 180°< θ <270°, or π< θ <3π/2. Finally, the fourth quadrant is defined by x>0, y<0, or 270°< θ <360°, or 3π/2< θ <2π.



4. Arc length and Area of Circle



Consider a circle of radius r and an angle of θ radians, as shown in the figure on the right. The sides of the angle θ radians determine an arc length s on the circle. We say that the ***central angle*** of θ radians ***intercepts an arc*** length s on the circle (or that arc length s is intercepted by a central angle of θ radians). It can be shown that the ratio of the arc length s to the circumference $2\pi r$ of the entire circle in the figure shown is the same as the ratio of the central angle of θ radians to the full – circle angle of 2π radians; that is,

Arch Length Formula: Length = $\frac{\theta}{360}2\pi$

$\frac{s}{2\pi r} = \frac{\theta}{2\pi}$, Solving this equation for s, we obtain the following fact.

A central angle of θ radians in a circle of radius r intercepts an arc of length $s = \theta r$, that is, the length s of the arc is the product of the radian measure of the angle and radius of the circle.

Example

In a circle of radius 8cm, find

a. The length of the arc which subtends an angle of $\frac{3\pi}{4}$ radians at the centre,

b. The angle subtended by an arc of length 6 cm.

Solutions:

a) $s = \theta r = 8\,\text{x}\frac{3\pi}{4} = 6\pi\,\text{cm}\,(\approx 18.8\text{cm})$

b) From the formula, $\theta = \frac{s}{r} = \frac{6}{8} = 0.75\,\text{rad}\,(= 0.75 \approx 43.0^{0})$

Unit: Trigonometry	Topic: Relationships and Sketching of Trigonometric Functions
Benchmark	**12.2.2.2.** Explain the relationship between trigonometric functions and their inverse and sketch the trigonometric functions.

Learning Objective: By the end of the topic, students will be able to;

- recognize the relationship between the trigonometric functions and their inverse functions.



Essential questions:

- What is a sine function?
- How can a sine function be graphed?
- What is the inverse function to the sine function?



Key Concepts(ASK-MT)	
Attitudes/Values	Enjoy how trigonometric functions and their inverse functions are related and how they defer in their graphs.
Skills	Sketching or Modelling of the graphs of trigonometric functions and their inverse functions
Knowledge	The relationships of trigonometric functions and their inverse functions and sketching the functions.
Mathematical Thinking	Think about the relationships between the trigonometric functions.

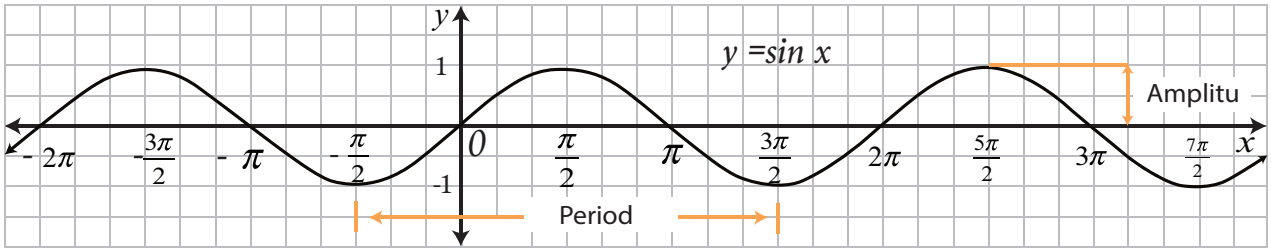
Content Background

1. Graph of the sine function

If the angle is expressed in radian measure then the trigonometric ratios may be defined as functions of a real variable.

For example, $\sin x$ is defined as the sine of an angle of size x radians and for every value of x there is one and only one value of $\sin x$.

The figure below shows the graph of $y = \sin x$ for $-2\pi \leq x \leq 2\pi$.



(i) Each time we increase or decrease the value of the x - coordinates by a multiple of 2π, the basic curve is repeated.

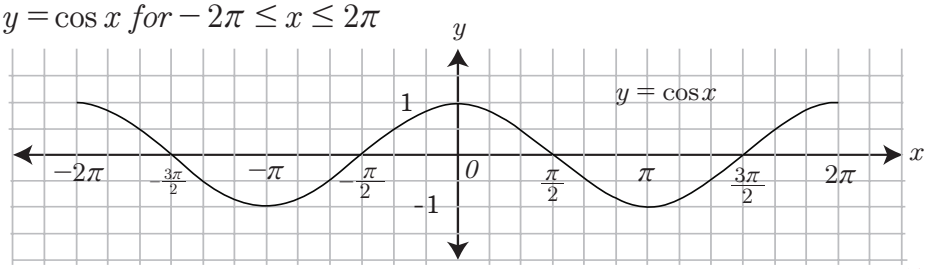
(ii) Each portion of the graph in an interval of 2π is one cycle of the sine function.

(iii) The **amplitude** of the curve $y = \sin x$ is 1.

(iv) The function $y = \sin x$ is called a ***periodic function*** with a ***period*** of 2π because for every x in the domain of the sine function, $\sin x = \sin(x + 2\pi)$.

2. The Graph of the Cosine function

The graph of $y = \cos x$ *for* $-2\pi \leq x \leq 2\pi$



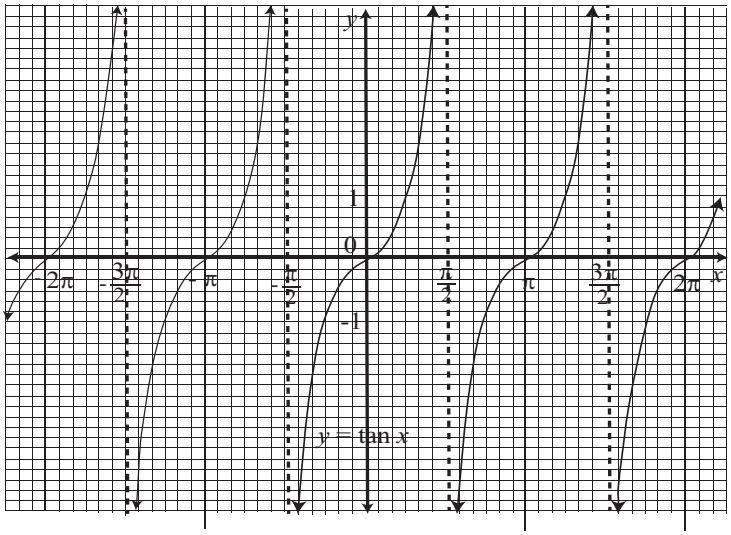
(i) The graph of $y = \cos x$ has the same wave shape as the graph of $y = \sin x$. In fact if the sine curve is translated a distance of $\frac{\pi}{2}$ to the left it falls on the cosine curve.

(ii) The period of $y = \cos x$ is 2π and the graph repeats at an interval of 2π.
$\cos x = \cos(x + 2\pi)$

(iii) $y = \cos x$ is an even function, $\cos(-x) = \cos x$, and its graph is symmetrical about the y axis.

3. The Graph of the Tangent Function

The graph of $y = \tan x \; for - 2\pi \leq x \leq 2\pi$.

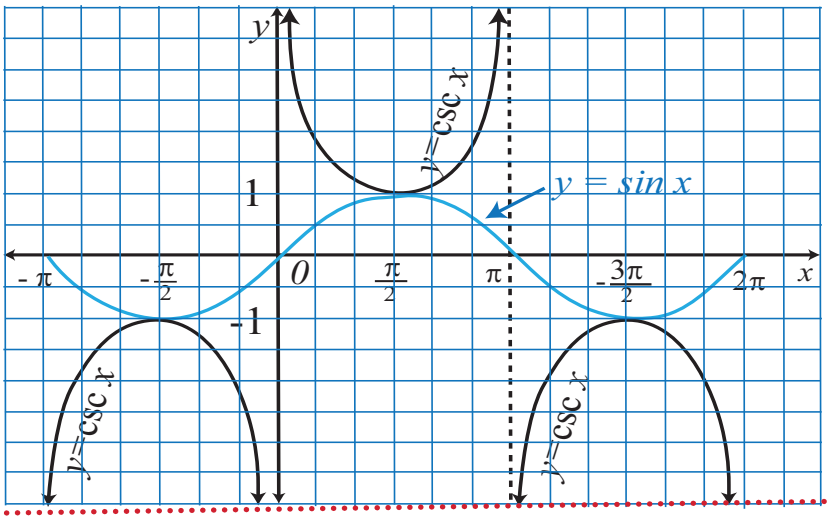


(i) Since $\tan x = \frac{\sin x}{\cos x}, \tan x$ is not defined for $\cos x = 0$. Hence the graph of $y = \tan x$ is discontinuous in this domain for $x = \frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$.

(ii) The period of is $y = \tan x$ is π and it can be seen that the graph repeats at interval of π. Hence $\tan x = \tan(x + \pi)$, .

4. The Graph of the Reciprocal Functions

(a) The graph of $y = \operatorname{cosec} x$ for $-\pi \leq x \leq 2\pi$. The graph of $y = \sin x$ is shown in blue. Since $\operatorname{cosec} x = \frac{1}{\sin x}$ the graph of $y = \operatorname{cosec} x$ can be sketched by using the reciprocal values of $\sin x$. We note that also the $\operatorname{cosec} x$ is not defined and its graph is discontinuous when $\sin x = 0$. That is, when $x = -\pi, 0, \pi, 2\pi$ the graph of $y = \operatorname{cosec} x$ has vertical **asymptotes**. The **range** of values for $y = \sin x \; is - 1 \leq y \leq 1$. The **range** of values for .The period of $y = \operatorname{cosec} x$ is 2π.

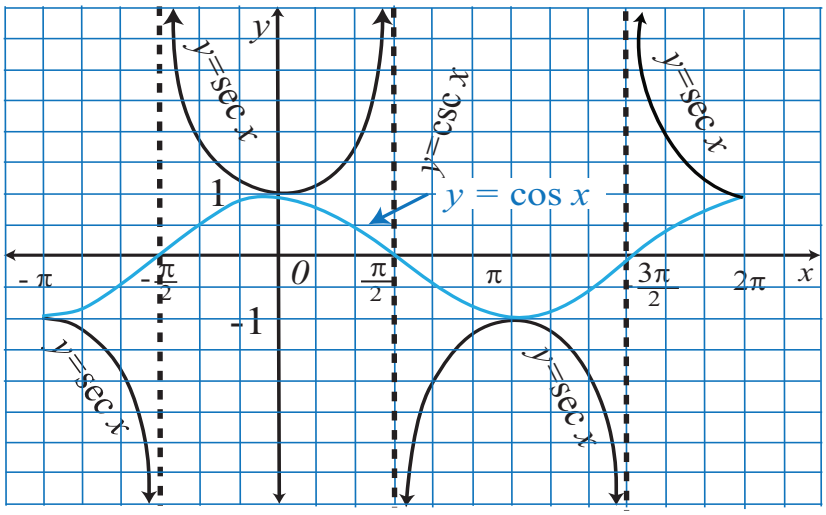


(b) The graph of $y = \sec x$ for $-\pi \leq x \leq 2\pi$. The graph of $y = \cos x$ is shown in blue.

(i) For what values of x is $\sec x$ not defined?

(ii) State the range for $y = \cos x$ and $y = \sec x$.

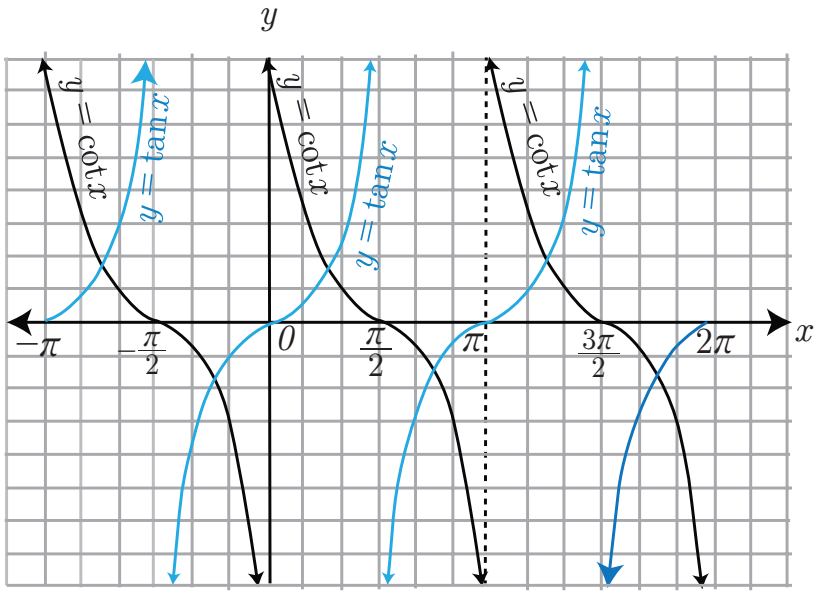
(iii) What is the period of the curve $y = \sec x$.



(c) The graph of $y = \cot x$ for $-\pi \leq x \leq 2\pi$. The graph of $y = \cot x$ for $-\pi \leq x \leq 2\pi$ is shown in blue.

(i) For what values of x is $\cot x$ not defined?

(ii) What is the period of the curve $y = \sec x$.



Unit: Trigonometry	Topic: Trigonometric Equations
Benchmark	**12.2.2.3** Find the value of a trigonometric function and inverse trigonometric function and solve trigonometric equations.

Learning Objective: By the end of the topic, students will be able to;

- solve trigonometric equations graphically.



Essential questions:

- What are trigonometric identities?
- How significant are the trigonometric identities?
- What are some common trigonometric identities?

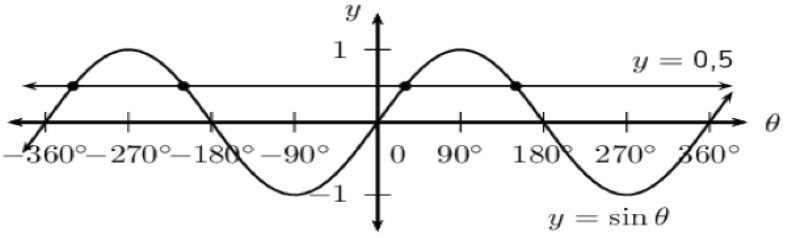


Key Concepts(ASK-MT)	
Attitudes/Values	Confidently solve trigonometric equations by using graphs.
Skills	Solving trigonometric equations.
Knowledge	Trigonometric equations.
Mathematical Thinking	Solve trigonometric equations graphically.

Content Background

Trigonometric equations

Equations such as $\sin\theta = \frac{1}{2}$ and $\cos\theta = -\frac{1}{3}$ are examples of trigonometric equations. Solving trigonometric equations requires that we find the value of the angles that satisfy the equation. If a specific interval for the solution is given, then we need only find the value of the angles within the given interval that satisfy the equation. If no interval is given, then we need to find the general solution. The periodic nature of trigonometric functions means that there are many values (infinite) that satisfy a given equation, as shown in the diagram below.



All possible values of the unknown which satisfy the given equation are called solutions of the given equation. For complete solution “all possible values” satisfying the equation must be obtained. When solving a trigonometric equation, you find all sets of values of θ which satisfy the given equation.

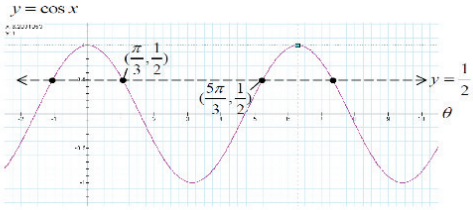
Equation Solving

Example 1: Find all possible values of θ so that

Solution: $\theta = \frac{3}{\pi} + 2\pi n$, $\theta = \frac{5\pi}{3} + 2\pi n$, where n is an integer.

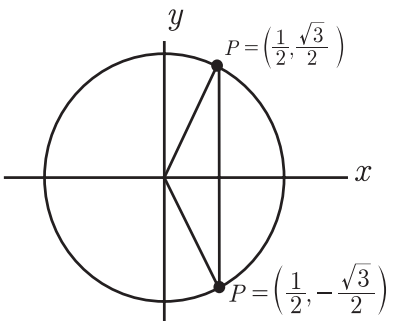
Solution Method #1 – Graphically

There are infinite number of solutions which are represented by the θ value of intersection points of the cosine curve and the constant function $y = \frac{1}{2}$.



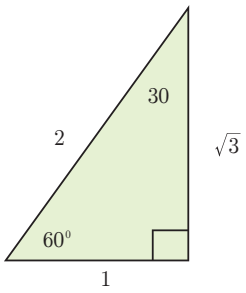
For $\theta \le \theta \le 2\pi$, there are two solutions: $\theta = \frac{\pi}{3}(60^0)$ and $\theta = \frac{5\pi}{3}(300^0)$. Generalizing, $\theta = \frac{\pi}{3} + 2\pi n, \theta = \frac{5\pi}{3} + 2\pi n$, where n is an integer. Thus all solutions differ from the original two solutions by multiples of the period of the cosine function.

Method # 2 - Unit Circle Approach



$\cos\theta = \frac{1}{2}$ occurs when $x = \frac{1}{2}$ for point(s) on the unit circle. The two points are $\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right)$. The corresponding angles are $\cos\theta = \frac{\pi}{3}$ and $\cos\theta = \frac{5\pi}{3}$. Generalizing, $0 = \frac{\pi}{3} + 2\pi n, \theta = \frac{5\pi}{3} + 2\pi n,$

Method # 3 - Unit Circle Approach



$\cos\theta = \frac{1}{2}$ is a special case that involves $30^0 - 60^0 - 90^0$ triangles. Since $\cos\theta = \frac{adj}{hyp} = \frac{1}{2}$, this implies θ must be 60^0 or $\frac{\pi}{3}$ radians.

Generalizing, $\cos\theta$ is positive in quadrant IV with a reference angle of 60º.

Generalizing completely, $\theta = \frac{\pi}{3} + 2\pi n, \theta = \frac{5\pi}{3} + 2\pi n,$

Example 1: Solve $2\sin\theta + 1 = 0, \text{for } 0^0 \le x \le 360^0$.

Solution: $2\sin\theta + 1 = 0, \text{so } \sin\ = -\frac{1}{2}$. The related angle is 30º because $\sin 30^0 = \frac{1}{2}$. Here, the sine is negative, so θ lies in the third or fourth quadrant. Hence, $\theta = 180^0 + 30^0 = 120^0 \text{or } \theta = 360^0 - 30^0 = 330^0$. The solutions in the given range are $\theta = 210^0 \text{and } \theta = 330^0$.

Example 2: Solve to the nearest degree, $5\cos\theta + 4, for\ 0^0 \le x \le 360^0$.

Solution: $5\cos\theta + 4 = 2, so\ \cos\theta = -\frac{2}{5} = -0.4$

From the calculator, the related angle is $\cos^{-1}0.4 \approx 66^0$, to the nearest degree. Since the cosine is negative, θ lies in the second or the third quadrant.

Hence, $\theta \approx 180^0 - 66^0 = 114^0 \text{ or } \theta \approx 180^0 + 66^0 = 246^0$.

Example 3: Solve $4\sin^2\theta = 1, \text{for } 0^0 \le x \le 360^0$.

Solution: $4\sin^2\theta = 1$

$\sin^2\theta = \frac{1}{4} \therefore \sin\theta = \frac{1}{2} \text{or } \sin\theta = -\frac{1}{2}$

The related angle is 30º, since $\sin 30^0 = \frac{1}{2}$. And since $\sin\theta$ is either positive or negative, the angle can be be in any one of the four quadrants, so $\theta = 30^0, 150^0, 210^0 \text{or } 330^0$.

Unit: Trigonometry	Topic: Trigonometric Identities and Equations
Benchmark	**12.2.2.4** Apply trigonometric identities to find solutions to trigonometric equations.

Learning Objective: By the end of the topic, students will be able to;



- applying the correct trigonometric identities to solve trigonometric equations, and
- appreciate the significance of trigonometric identities in solving trigonometric equations.

Essential questions:



- What are trigonometric identities?
- How significant are the trigonometric identities?
- What are some common trigonometric identities?

Key Concepts(ASK-MT)	
Attitudes/Values	Confidently apply the appropriate trigonometric identities to solve trigonometric equations.
Skills	Solving trigonometric equations.
Knowledge	Trigonometric identities and equations.
Mathematical Thinking	Think about how to identify the correct trigonometric identities to solve trigonometric equations.

Content Background

1. Fundamental Identities

The ***Reciprocal Identities***

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

The ***Ratio/Quotient Identities***

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Even-Odd Identities

The cosine and secant functions are even.

$$Cos(-t) = \cos t \qquad \sec(-t) = \sec t$$

The sine, cosecant, tangent, and cotangent functions are odd.

$$\sin(-t) = \sin t \qquad \csc(-t) = -\csc t$$
$$\tan(-t) = -\tan t \qquad \cot(-t) = -\cot t$$

The ***Pythagorean Identities***

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \; 1 + \cot^2\theta = \csc^2\theta$$
$$\sin^2\theta = 1 - \cos^2\theta \qquad \cos^2\theta = 1 - \sin^2\theta$$

1. Fundamental Identities

The ***Reciprocal Identities***

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Even-Odd Identities

The cosine and secant functions are even.

$$Cos(-t) = \cos t \qquad \sec(-t) = \sec t$$

The sine, cosecant, tangent, and cotangent functions are odd.

$$\sin(-t) = \sin t \qquad \csc(-t) = -\csc t$$
$$\tan(-t) = -\tan t \qquad \cot(-t) = -\cot t$$

The ***Pythagorean Identities***

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \; 1 + \cot^2\theta = \csc^2\theta$$
$$\sin^2\theta = 1 - \cos^2\theta \qquad \cos^2\theta = 1 - \sin^2\theta$$

2. Using Fundamental Identities to Verify Other Identities

The fundamental trigonometric identities are used to establish other relationships among trigonometric functions. To verify an identity we show that one side of the identity can be simplified so that is identical to the other side. Each side is manipulated independently of the other side of the equation. Usually it is best to start with the more complicated side of the identity first.

Example 1: Verify the identity: $\sec x \cot x = \csc x$.

Solution: The left side of the equation contains the more complicated expression. Thus, we work with the left side. Let us express this side of the identity in terms of sines and cosines. Perhaps this strategy will enable us to transform the left side into csc x, the expression on the right.

$$\sec x \cot x = \frac{1}{\cos x} \cdot \frac{\cos x}{\sin x}$$
$$= \frac{1}{\sin x}$$
$$= \csc x$$

Example 2: Verify the identity: $\sin x \tan x + \cos x = \sec x$.

Solution: We start with more complicated side, the left side.

$$\sin x \tan x + \cos x = \sin x \frac{\sin \boldsymbol{x}}{\cos \boldsymbol{x}} + \cos x = \frac{\sin \boldsymbol{x} \sin \boldsymbol{x} + \cos \boldsymbol{x} \cos \boldsymbol{x}}{\cos x} = \frac{\sin^2 x + \cos^2 x}{\cos x} = \frac{1}{\cos x} = \sec x$$

Unit: Trigonometry	Topic: Use of Trigonometric Identities
Benchmark	**12.2.2.5** Use fundamental trigonometric identities, including sum and difference, double-angle and half angle formulas to solve problems.

Learning Objective: By the end of the topic, students will be able to;

- apply the sum and differences of the trigonometric identities to solve trigonometric problems,
- recognise and apply the double-angle identities to problem solving, and
- apply the half-angle formula to solve trigonometric problems.



Essential questions:

- What is the sum and difference of the trigonometric identities?
- What is a double-angle formula in the identities?
- What is the half-angle formula?



Key Concepts(ASK-MT)	
Attitudes/Values	Recognising and applying the correct identities to solve trigonometric problems either by using the sum and difference, half-angle or double-angle formulae.
Skills	Solve trigonometric equations using trigonometric identities.
Knowledge	Sum and differences, double-angle and half-angle formula.
Mathematical Thinking	Think about how to apply the correct formula to solve trigonometric equations.

Content Background

Sum and Difference Formulas

In this topic we will use formulas that involve the sum or difference of two angles, call the sum and difference formulas.

1. Sum and difference formulas for sines and cosines

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

How do we use these formulas?

Example (i) Find the exact value of $\cos 105^0$

Solution: We can break 105^0 into 60^0 and 45^0 since these values are relatively easy to find the cosine of.

Therefore $\cos 105^0 = \cos(60^0 + 45^0) = \cos 60^0 \cos 45^0 - \sin 60^0 \sin 45^0$

Using the unit circle we obtain,

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{1}{4}(\sqrt{2} - \sqrt{6}$$

Example (ii) Find the exact value of $\sin 15^0$

$$= \sin(45^0 - 30^0)$$
$$= \sin 45^0 \cos 30^0 - \cos 45^0 \sin 30^0$$
$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$
$$= \frac{1}{4}(\sqrt{6} - \sqrt{2})$$

2. Sum and difference formulas for tangent

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Example (iii) Find the exact value of $\tan 75^0$

Solution: $\tan 75^0 = \tan(45^0 + 30^0)$

$$= \frac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}} = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} \text{.....(by rationalizing the denominator)}$$
$$= \frac{3 + \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}}$$
$$= 2 + \sqrt{3}$$

Example (iv) Find $\tan \frac{7\pi}{12}$.

Solution: $\tan \frac{7\pi}{12} = \tan\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right) = \tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$

$$= \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{3}}{1 - \tan\frac{\pi}{4} \cdot \tan\frac{\pi}{3}} = \frac{1 + \sqrt{3}}{1 - \sqrt{3}} = \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3}$$

3. Cofunction Identities

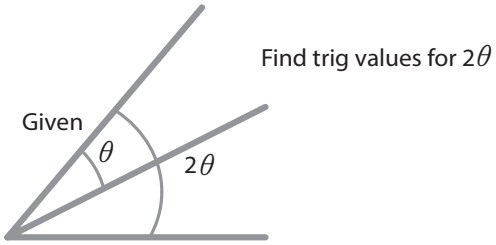
$$\cos(90^0 - \theta) = \sin\theta \qquad \sec(90^0 - \ ) = \csc$$
$$\sin(90^0 - \theta) = \cos\theta \qquad \csc(90^0 - \theta) = \sec\theta$$
$$\tan(90^0 - \theta) = \cot\theta \qquad \cot(90^0 - \theta) = \tan\theta$$

Example (v): Find $\cos 30^0$

Solution: $\cos 30^0 = \sin(90^0 - 30^0) = \sin 60^0 = \frac{\sqrt{3}}{2}$

4. Double Angle Formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$



5. Half Angle Formulas

$$\cos\frac{\theta}{2} = \pm\frac{\sqrt{1+\cos\theta}}{2}$$
$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \qquad \tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} \qquad \tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$$

Example (vi): Use the double angle formula to find the exact value of $\sin 120^0$

Solution: $\sin 120^0 = \sin 2(60^0) = 2\sin\cos 60^0 = \frac{\sqrt{3}}{2}$

Example (vii) $\tan\theta = \frac{5}{12}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\cos 2\theta$.

Solution: First we need to find what the $\cos\theta$ is. We know that $\tan\theta$ is the opposite leg over adjacent leg, so we need to find the hypotenuse since cos is adjacent over hypotenuse. Now can use $r = \sqrt{12^2+5^2} = 13$ to find the length of the hypotenuse. Now we know that $\cos\theta = \frac{12}{13}$.

Now use the double angle formula to find $\cos 2\theta$.

$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{5}{13}\right)^2 = 1 - 2\frac{25}{169} = 1 - \frac{50}{169} = \frac{119}{169}$$

We take the positive answer since θ is in the third quadrant making the ratio a negative over a negative.

Example (viii) Use the half angle formula to find the exact value of $\cos 15^0$

Solution: $\cos 15^0 = \cos\frac{30}{2} = \pm\sqrt{\frac{1+\cos 30}{2}} = \pm\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}$

Choose the positive root.

Example (ix): $\cos\theta = -\frac{4}{5}$ *and* $90^0 < \theta < 180^0$. *Find* $\sin\frac{\theta}{2}$.

Solution: First we use the Pythagorean Theorem to find the third side $4^2 + x^2 = 5^2$

Then, $x^2 = 9 \therefore x = 3$

$$\sin\theta = \pm\sqrt{\frac{1-\left(-\frac{4}{5}\right)}{2}} = \pm\sqrt{\frac{\frac{9}{5}}{2}} = \pm\sqrt{\frac{9}{10}} = \frac{3\sqrt{10}}{10}$$

Since sine is positive in the second quadrant we take the positive answer.

Unit: Trigonometry	Topic: Verification of Trigonometric Identities
Benchmark	**12.2.2.6** Verify trigonometric identities.

Learning Objective: By the end of the topic, students will be able to;

- explain and Verify the trigonometric identities.



Essential questions:

- How can trigonometric identities be verified?



Key Concepts(ASK-MT)	
Attitudes/Values	Confidently providing reasons or explanations to verify trigonometric identities.
Skills	Analytically verify trigonometric identities.
Knowledge	Trigonometric identities.
Mathematical Thinking	Think about how to verify trigonometric identities.

Content Background

You can use the fundamental identities to verify new trigonometric identities. A verification of an identity is a chain of equivalent expressions showing that one side of the identity is equal to the other side. When verifying an identity, begin with the expression from one side and manipulate it algebraically until it is identical to the other side. Sometimes you will have to work on both sides and show at some stage that they are the same.

Examples

(i) Verify the identity: $\frac{\cot^2 x}{\csc x} = \csc x - \sin x$

Solution

$$\frac{\cot^2 x}{\csc x} = \frac{\csc^2 x - 1}{\csc x} \quad \text{..... Pythagorean Identiy}$$

$$= \frac{\csc^2 x}{\csc x} - \frac{1}{\csc x} \quad \text{.....Write as separate fractions}$$

$$= \csc x - \frac{1}{\csc x} \quad \text{..... Simplify}$$

$$= \csc x - \sin x \quad \text{.....Reciprocal Identity}$$

(ii) Verify the identity:

Solution

$$\frac{\sin x}{1-\cos x} = \frac{\sin x(1+\cos x)}{(1-\cos x)(1+\cos x)} \quad \text{.....Multiply by} \frac{1+\cos x}{1+\cos x}$$

$$= \frac{\sin x(1+\cos x)}{1-\cos^2 x} \quad \text{.....Simplify Denominator}$$

$$= \frac{\sin x(1+\cos x)}{\sin^2 x} \quad \text{.....Pythagorean identity}$$

$$= \frac{1+\cos x}{\sin x} \quad \text{..... Simplify}$$

(iii) Verify the identity: $\sin\theta\cot\theta = \cos\theta$

Solution: To prove this identity we transform the left side into the right side:

$$\begin{aligned}\sin\theta &= \sin\theta.\frac{\cos\theta}{\sin\theta} \quad \text{.....Ratio}\\ &= \frac{\sin\theta\cos\theta}{\sin\theta} \quad \text{.....Multiply}\\ &= \cos\theta \quad \text{.....Divide out common factor.}\end{aligned}$$

(iv) Prove: $\tan x + \cos x = \sin x(\sec x + \cot x)$.

Solution: We begin by applying the distributive property to right side of the identity. Then we change each expression on the right side to an equivalent expression involving only $\sin x$ and $\sin x$ and $\cos x$.

$$\begin{aligned}\sin x(\sec + \cot x) &= \sin x\sec x + \sin x\cot x \quad \text{.....Multiply}\\ &= \sin x.\frac{1}{\cos x} + \sin x.\frac{\cos x}{\sin x} \quad \text{.....Reciprocal and ratio identity}\\ &= \frac{\sin x}{\cos x} + \cos x \quad \text{.....Multiply and divide out common factor } \sin x\\ &= \tan x + \cos x \quad \text{.....Ratio identitu}\end{aligned}$$

In this case, we have transformed the right side into the left side.

(iv) Prove: $\dfrac{\cos^4 t - \sin^4 t}{\cos^2 t} = 1 - \tan^2 t$

Solution: In this example, factoring the numerator on the left side will reduce the exponents there from 4 to 2.

$$\begin{aligned}\frac{\cos^4 t - \sin^4}{\cos^2 t} &= \frac{(\cos^2 t + \sin^2 t)(\cos^2 t - \sin^2 t)}{\cos^2 t} \quad \text{.....Factor}\\ &= \frac{1(\cos^2 t - \sin^2 t)}{\cos^2 t} \quad \text{..... Pythagorean Identity}\\ &= \frac{\cos^2 t}{\cos^2 t} - \frac{\sin^2 t}{\cos^2 t} \quad \text{.....Seperate into two identities}\\ &= 1 - \tan^2 t \quad \text{.....Pythagorean identity}\end{aligned}$$

(v) Prove: $\sin 3x = 3\sin x - 4\sin^3 x$

$$\begin{aligned}\sin 3x &= \sin(2x + x)\\ &= \sin 2x\cos x + \cos 2x\sin x \quad \text{.....using the first addition formula}\\ &= (2\sin x\cos x)\cos + (1 - 2\sin^2 x)\sin x \quad \text{.....using the double angle formula } \cos 2x = 1 - 2\sin^2 x\\ &= 2\sin x\cos^2 x + \sin x - 2\sin^3 x\\ &= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x \quad \text{.....from the identity } \cos^2 x + \sin^2 x = 1\\ &= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x\\ &= 3\sin x - 4\sin^3 x\end{aligned}$$

Unit: Trigonometry	Topic: Trigonometric Equations with and without restricted domains
Benchmark	**12.2.2.7** Solve trigonometric equations and inverse trigonometric equations that include all solutions or solutions with restricted domains.

Learning Objective: By the end of the topic, students will be able to;

- evaluate trigonometric equations with real solutions, and
- evaluate trigonometric equations with restricted domain.

Essential questions:

- Do all the trigonometric equations have real roots/solutions?
- What does it mean to have solutions with restricted domains?

Key Concepts(ASK-MT)	
Attitudes/Values	Show confident in solving trigonometric equations with real solutions and those with restricted domain.
Skills	Analysis of trigonometric equations with real solutions and those with restricted domain.
Knowledge	Trigonometric equations with and without restricted domains.
Mathematical Thinking	Think about how to solve trigonometric equations with and without restricted domains including inverse trigonometric equations.

Content Background

1. Solutions of Trigonometry Without Restrictions

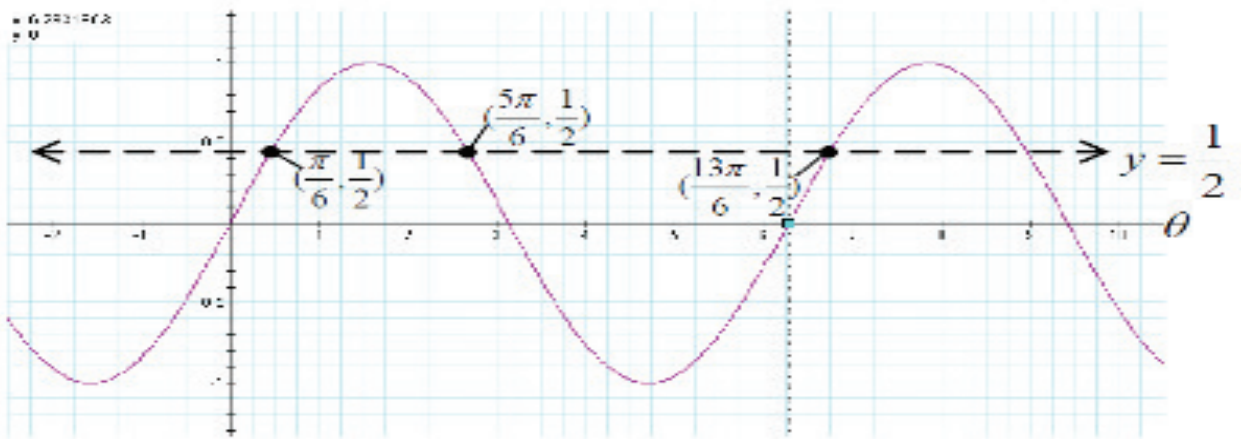
Example: Find 3 positive and 2 negative solutions for $\sin\theta = \frac{1}{2}$

Solution: There are many different correct solutions. One solution set is

$$\theta = \left\{-\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}\right\}$$

Graphically, five solutions are the θ values of the 5 points of the intersection of the sine curve and horizontal line $y = \frac{1}{2}$ shown on the right.
$y = \sin x$ (3 positive solutions are depicted in the graph)

Thus, $\theta = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$.



2. Solutions of Trigonometry with Restrictions

Example: **(i)** Solve for : $\sqrt{3}\sin x - 2\sin x\cos x = 0$ *for* $\leq x < 2\pi$.

Solution: Factor the expressions on the left and set each factor to zero.

$$\sqrt{3}\sin x - 2\sin x\cos x = 0$$

$$(\sin x)(\sqrt{3} - 2\cos x) = 0 \therefore \sin x = 0 \text{ or } \sqrt{3} - 2\cos x = 0$$

$$x = 0, \pi \qquad \cos x = \frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{6}, \frac{11\pi}{6}$$

Answers: $x = 0, \frac{\pi}{6}, \pi, \frac{11\pi}{6}$

(ii) Solve for θ in each equation. Assume $0 \leq \theta \leq 2\pi$.

Solutions: a. $\sin\theta = -\frac{\sqrt{3}}{2}$ b. $\cos\theta = 1$ c. $\tan\theta = 1$

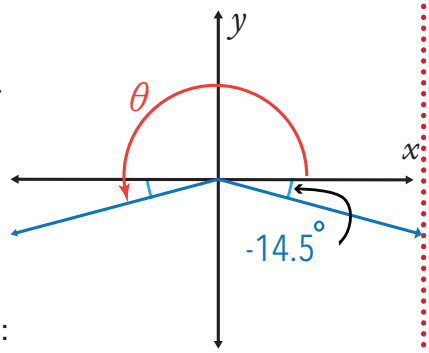
a. To solve the equation $\theta = -\frac{\sqrt{3}}{2}$, first remember that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Because sine is negative in the third and fourth quadrants, it follows that you are seeking values of θ in these quadrants that have a reference angle of $\frac{\pi}{3}$. The two angles fitting these criteria are $\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ *and* $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

3. Solutions of Trigonometry Equations.

Example: Solve the equation $\sin\theta = -\frac{1}{4}$ where $180^0 < \theta < 270^0$.

Solution: In the interval $-90^0 < \theta < 90^0$, the angle sine *is* $-\frac{1}{4}$ *is* $\sin^{-1}\left(-\frac{1}{4}\right) \approx -14.5^0$.



This angle is in Quadrant IV as shown. In Quadrant III (where $180^0 < \theta < 270^0$) the angle that has the same sine value is:

$$\theta \approx 180^0 + 14.5^0 = 194.5^0$$

Use a calculator to check the answer.

$$\sin 194.5^0 \approx -0.25 \text{ correct}$$

Unit: Trigonometry	Topic: Graphs of Sine and Cosine Functions
Benchmark	**12.2.2.8** Use the trigonometric functions in the form $y = A\sin(Bx + C) + D$ and $y = A\cos(Bx + C) + D$ to determine various properties of the function including domain, range, period, phase shift and amplitude.

Learning Objective: By the end of the topic, students will be able to;

- determine domain, range, period, phase shift and amplitude using the trigonometric function $y = A\sin(Bx + C) + D$ and $y = ACos(Bx + C) + D$.

Essential questions:

- How do these parameters A, B, C, and D affect the shape of the graph $y = A\sin(Bx + C) + D$?
- Why are these parameters A, B, C and D important?

Key Concepts(ASK-MT)	
Attitudes/Values	Confidently define and illustrate the properties to the trigonometric functions.
Skills	Analysing the properties to the trigonometric functions.
Knowledge	Domain, range, period, phase shift and amplitude.
Mathematical Thinking	Think about and evaluate domain, range, period, phase shift and amplitude using the trigonometric equation $y = A\sin(Bx + C) + D$ and $y = A\cos(Bx + C) + D$.

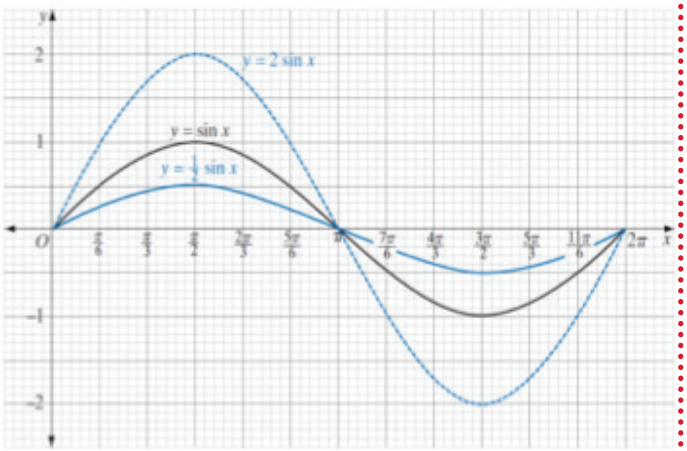
Content Background

Key features of the sine and cosine functions

Functions such as $af(x), f(ax)$ and $f(x) + a$ are transformations of the function $f(x)$. Each of these transformations can be applied to the sine function and cosine function.

1. Amplitude

How do the functions $y = 2\sin x$ and $y = \frac{1}{2}\sin x$ compare with the function $y = \sin x$?



We see that $y = 2\sin x$ is the function $y = \sin x$ stretched in the vertical direction and $y = \frac{1}{2}\sin x$ is the function $y = \sin x$ compressed in the vertical direction as expected.

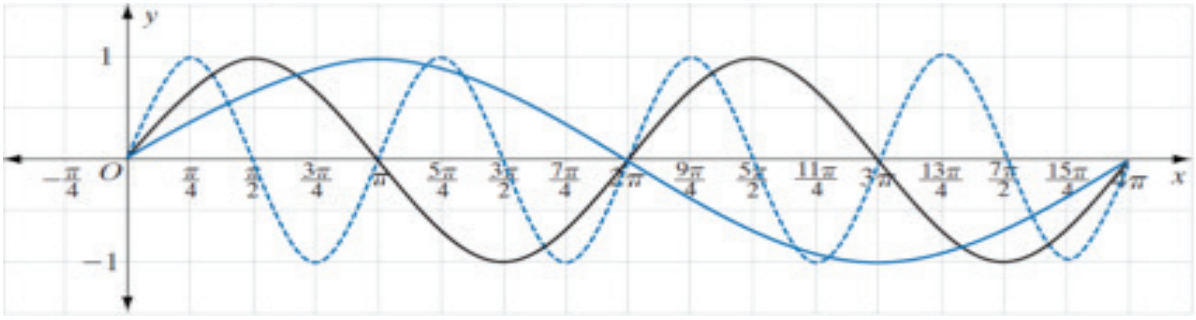
Amplitude measures how many units above and below the midline of the graph the function goes. Thus, for $y = \sin x$, the amplitude is 1, for $y = 2\sin x$, the amplitude is 2 and for $y = \frac{1}{2}\sin x$, the amplitude is $\frac{1}{2}$. In general, for $y = a\sin x$ and $y = a\cos x$, the amplitude is $|a|$.

2. Period

The function $g(x) = f(ax)$ is the function $f(x)$ stretched or compressed by a factor of a in the horizontal direction.

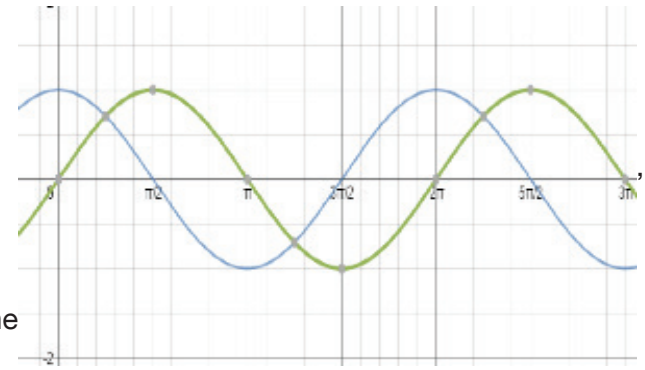
Compare the graphs of $y = \sin x, y = \sin 2x$ and $y = \sin\frac{1}{2}x$. The difference between the

x – coordinates of the endpoints of the interval for one cycle of the graph is the **period** of the graph. The period of $y = \sin x$ *is* 2π, the period of $y = \sin 2x$ is π and the period of $y = \sin \frac{1}{2}x$ is 4π. In general, the period of $y = \sin bx$ and $y = \cos bx$ is $|\frac{2\pi}{b}|$.



3. Phase Shift

Not only can the graphs be contracted or stretched vertically and horizontally, but they can also be shifted left and right and up and down. First we will focus on shifting the graph left and right. Since the sine function has been getting all the action, let's look at the cosine function $y = \cos x$. If we change the equation to $y = \cos\left(x - \frac{\pi}{2}\right)$ watch what
happens to the graph.

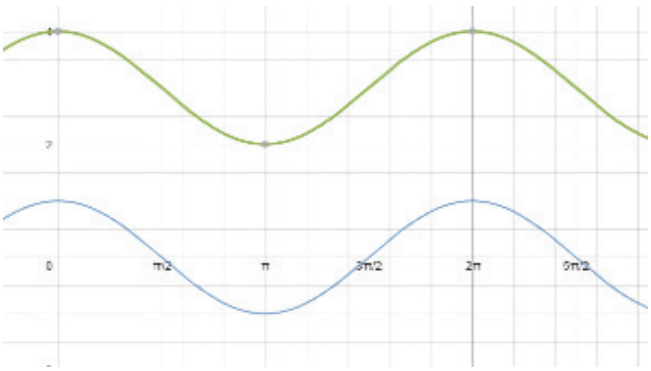


The blue wave is the original $y = \cos x$ and the green wave is the $y = \cos\left(x - \frac{\pi}{2}\right)$. You will notice that the graph was shifted $\frac{\pi}{2}$ units to the right. If the equation had been , the graph would have shifted $\frac{\pi}{2}$ units to the left. When the graph is shifted to the left or right it is called a "phase shift". If you consider the equation $y = a\cos(bx - c)$, the phase shift can be found by taking $\frac{c}{b}$.

Again the sine function is affected the same.

4. Vertical Shift

The last way we can alter the sine and cosine functions is by making a vertical shift. Let's take a look at what happens to the function when we change it to $y = \cos x + 3$ (Notice the 3 is not in parenthesis)



The blue wave is the original $y = \cos x$ and the green wave is the function$y = \cos x + 3$. When you add a number at the end you shift the graph up that many units and if you subtract a number at the end you shift the graph down that many units. When the function is written in the form $y = a\cos(bx - c) + d$, ***d*** controls whether the function will be shifted up or down.

Example

State the amplitude, period, phase shift, and vertical shift for each function.

(i) $y = \frac{1}{2}\sin\left(\theta - \frac{\pi}{2}\right) + 4$ ii $y = \cos 3\ \theta - 45^0\ \frac{1}{2}$

Solution

Given: $a = \frac{1}{2}, b = 1, c = \frac{\pi}{2} and\ \mathrm{d} = 4$

Amplitude: $a\ = |\ \frac{1}{2}\ | = \frac{1}{2}$ Amplitude: $|a\ | = |\ 1\ | = 1$

Period: $\frac{2\pi}{b} = |\ \frac{1}{2}\ | = \frac{1}{2}$

Phase Shift: $\frac{c}{b} = \frac{\pi}{2}$ Phase Shift: $\frac{c}{b} = 45^0$

Vertical Shift: $d = 4$ Vertical Shift: $d = \frac{1}{2}$

Unit: Trigonometry	Topic: Authentic Problems with Sine Functions
Benchmark	**12.2.2.9** Identify real-world phenomena that can be represented by the0 trigonometric function in the form $y = A\sin(Bx + C) + D$.

Learning Objective: By the end of the topic, students will be able to;



- analyse and solve real world phenomena represented by the trigonometric function in the form of $y = A\sin(Bx + C) + D$.

Essential questions:



- Can the trigonometric function $y = A\sin(Bx + C) + D$ be represented in some real world situations?

Key Concepts(ASK-MT)	
Attitudes/Values	Appreciate how real life problems can be described using trigonometric functions and displaying it on a graph.
Skills	Logically relate real life problems to trigonometric functions.
Knowledge	Real world phenomena and the function $y = A\sin(Bx + C) + D$.
Mathematical Thinking	Analysing real world phenomena relating to the trigonometric function $y = A\sin(Bx + C) + D$.

Content Background

Real-World Phenomena

One of the important features of periodic functions is their ability to model various real world applications like electrical currents, pendulums, blood pressure, temperature variations and even the wings of insects over time.

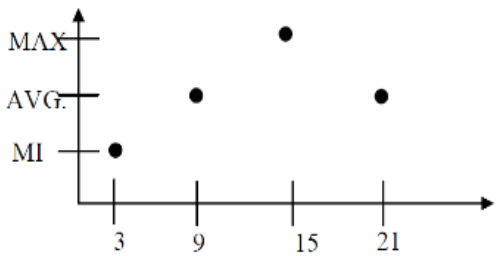
Example 1

Scientists sometimes use the formula $f(t) = a\sin(bt + c) + d$ to simulate temperature variations during the day, with time t in hours, temperature $f(t)$ in ^{0}C, and $t = 0$ corresponding to midnight, Assume that $f(t)$ is decreasing at midnight.

a) Determine the values of a, b, c and d that fit the information. Note: The graph of this curve will only be to the right of the $y - axis$. Therefore the value of c will be negative, since the phase shift will be to the right.

b) Sketch the graph of f for $0 \leq t \leq 24$.

1) The temperature varies between 10°C and 30°C, and the average temperature of 20°C **first occurs** at **9 AM**.



Period = 24 hours, 'hash marks' would be every 6 hours. Average is at 9 AM, 6 hours earlier (3 AM) would be minimum.

Since the average is at 20, the graph has been shifted up 20. d = 20
The difference between the maximum of 30 and the average or the minimum of 10 and the average is 10°. amplitude = a = 10

Period = $24\frac{2\pi}{} \rightarrow 24 \ = 2\pi \rightarrow \ = \frac{2\pi}{12}$

Phase shift is where the curve is rising at the average:

$$9 = -\frac{c}{\frac{\pi}{12}} \rightarrow 9\pi = -12c \rightarrow c = -\frac{3\pi}{4}$$

2) The high temperature of 28°C occurs at 2 PM and the average temperature of 20°C occurs 6 hours later.

Strand 3: Patterns and Algebra

Content Standard: Students will be able to interpret various types of patterns and functional relationships, use symbolic forms to represent, model, and analyze mathematical situations and collect, organise, and represent data to answer questions.

Units	Benchmark	Topics	Suggested Lesson Titles
Sequence and Series	**12.3.3.1** Differentiate between the nth term and sum of n terms of arithmetic and geometric sequences and use of sigma notation.	The n^{th} term and sum of n terms of the Arithmetic Sequence	The nth term of the Arithmetic Sequence
			Sum of n terms in a Arithmetic Sequence
			Use Sigma Notation
		The nth term Sum of n terms in Geometric Sequences	The nth term of the Geometric Sequence
			Sum of n terms in a Geometric Sequence
			Use Sigma Notation
	12.3.3.2 Apply the properties of arithmetic and geometric sequences and series to solve problems.	Properties of Arithmetic and Geometric Sequences and series	Solve problems using the properties of Arithmetic Sequence
			Solve problems using the properties of Geometric Sequence
		Infinite Geometric Series	Limiting sum of geometric series
Binomial Theorem	**12.3.3.3** Use the Pascal's triangle and the binomial theorem to expand binomial expression.	Pascal's Triangle	Use Pascal's triangle to expand binomial expressions
		Factorial, Permutation and Combination	Factorial and Combination
			Permutation
		Binomial Theorem	Introduction to Binomial Theorem and its properties
			The coefficient of the nth term in the binomial expression
			The nth term of the binomial expression
			Problems involving Binomial Theorem
Determinants	**12.3.3.4** Solve systems of linear equations in two or three variables using determinants.	Determinants	Introduction to determinants
			Determinants of a 2 by 2 system
			Determinants of a 3 by 3 system
			Solutions to systems of linear equations using determinants

Limits	**12.3.3.5** Recognize limits from graphs and tables.	Finding Limits numerically and graphically	Definition of limits and its Numerical representation
			Graphical representation of limits
	12.3.3.6 Find limits of sums, differences, products, quotients and rational functions.	Finding limits algebraically using the Limit laws	Properties of Limits
			Addition and Constant Theorems
			Multiplication and Division Theorems
			Limits of Rational Functions
	12.3.3.7 Define continuity in terms of limits and functions.	Continuity and Limits	Definition of Continuity and its properties
			Continuous Functions
			Limits involving infinity
Differentiation	**12.3.3.8** Calculate the derivatives of functions, including polynomial, rational, trigonometric, logarithmic, inverse, composite and exponential functions	The Gradient Function	Introduction to Differentiation
			The Limiting Case of the Secant Line and the Tangent to a Curve
		Definition of Derivatives from First Principle	Secant and Tangent line and Derivative at a point
			Using the definition of Derivatives
			Gradient and Equation of Tangent lines
		Standard Rules of Differentiation	Standard Rules of Differentiation
			Use of the Standard Rules of Differentiation
		General Rules of Differentiation	Product and Quotient Rules
			Function of a Function Rules
			Derivative of Rational Functions
			The Chain Rule for Composite Functions
		Derivatives of Exponential and Logarithm Functions	Derivative of Exponential Functions
			Derivatives Logarithm Functions
		Differentiation of the trigonometric functions	Derivatives of Trigonometric Functions and their Reciprocals
		Implicit Differentiation	Finding Derivatives of Relations and Functions Using Implicit differentiation method.
			Derivatives of Exponential and Logarithmic functions

	12.3.3.9 Calculate tangents, normal and stationary points on curves.	Maxima, minima, tangents, normal and stationary points	Normals and Tangents
			Stationary Points, Maximum and Minimum turning Points
	12.3.3.10 Solve problems on maxima and minima, and practical problems on rate of change.	The Second Derivative	Second Derivative Test for Maximum or Minimum and Increasing or Decreasing
			Curve Sketching
		Applied Optimisation Problems	Optimisation - Application of maximum and minimum ideas
Integration	**12.3.3.11** Define theorems and rules of integrations	Definition of Area under the Curve	Introduction to Integration
			Constant, sum and differences theorems and constant of integration
		Rules of integration	The Power rule
	12.3.3.12 Calculate the anti-derivatives of functions, including polynomial, rational, trigonometric, inverse, composite and exponential functions.	Anti-Derivatives and Indefinite Integrals	Finding the anti-derivatives
			Indefinite integral
		Integrals of Trig. And Exponential Functions	Integrals of derivatives of standard trigonometric functions, $\sin\theta$, $\cos\theta$ and $\tan\theta$
			Integrals of exponentials functions
			Evaluating Definite Integrals
	12.3.3.13 Solve problems on applications of integration.	Applications – Area of Curves	Area bounded by a line and the axes,
			Area bounded by a line and a curve, bounded by curves
			Area bounded by two curves

Unit: Sequence and Series	Topic : The n^{th} term and sum of n terms of the Arithmetic Sequence
Benchmark	**12.3.3.1** Differentiate between the n^{th} term and sum of n terms of the arithmetic and geometric sequences and use of the sigma notation.

Learning Objective: By the end of the topic, students will be able to;



- write the n^{th} term expression or formula for sequences,
- use the arithmetic sequence and formulas to derive the nth term and the sum of n terms,
- understand the meaning of the use of sigma notation and apply it, and
- use sum of n terms of AP formula to solve problems involving summation (sigma).

Essential questions:



- What is an n^{th} term?
- What is an arithmetic sequence?
- What is the meaning and use of sigma notation?

Key Concepts(ASK-MT)	
Attitudes/Values	Show interest to identify the patterns and evaluating the n^{th} term in the arithmetic sequences.
Skills	Sequencing and evaluating n^{th} terms in arithmetic sequences.
Knowledge	Arithmetic Sequences and the n^{th} term.
Mathematical Thinking	Think about how to use the n^{th} term, sigma notation and solving arithmetic sequence problems.

Content Background

1. Definition of Sequence and Term

If a set of numbers is written down in order according to some rule or pattern, the set of numbers is called a sequence and each number is called a term of the sequence. In some sequences the pattern can readily be seen and this allows us to predict subsequent terms in the sequence. For example consider the sequence 1, 3, 6, 10, 15, 21, 28 *etc*… The difference between the successive terms increases by 1 each time, so the sequence may be continued indefinitely. The next three terms in the sequence are 36, 45 and 55.

2. General Term or the n^{th} Term

It is possible to indicate a sequence by giving its general term. The symbol T_n is used to represent the general term or the n^{th} term of a sequence. *For example*, if $T_n = 3n + 2$, then the first term is found by substituting 1 for n in this expression and so on for the other terms:

Thus $T_1 = 3 \times 1 + 2 = 5, T_2 = 3 \times 2 + 2 = 8, T_3 = 3 \times 3 + 2 = 11$ giving the sequence, 5, 8, 11… We would say this is the sequence whose general term is $3n + 2$.

3. Arithmetic Sequence

An *arithmetic sequence or arithmetic progression* is a sequence in which each term after the first is formed by adding a constant number to the preceding term. The sequence 6, 10, 14, 18, 22, 26 … for example, progresses by adding the constant number 4. This sequence is an arithmetic sequence and the constant number added is called the *common difference* of the sequence.

4. The nth term of an arithmetic sequence

The nth term of arithmetic sequence may be found by letting a representing the first term and d the common difference. The sequence is then: $a, a+d, a+2d, a+3d, a+4d$..... The nth term of this sequence may be obtained by examining the table below.

T_1	T_2	T_3	T_4	T_5	.....	T_n
a	a + d	a + 2d	a + 3d	a + 4d	.....	a + (n-1)d

The coefficient of d in each term is one less than the number of the term. Thus the nth term of the arithmetic sequence $a, a+b, a+2d, a+3d$.....*is given by* $\mathrm{T_n} = a+(n-1)d$.... is given by $T\mathrm{n} = a+(n-1)d$.

5. The sum to *n* terms

The sum of the first n terms of arithmetic sequence given the first term a and last term l is given by the formula $\mathrm{S_n} = \frac{}{2}[a+1]$.

The sum of the first n terms of arithmetic sequence given the first term a and common difference d is given by the formula $S_n = \frac{n}{2}[2a+(n-1)d]$

6. Use of Σ (Sigma) Notation

Sigma notation is a shorthand method for indicating the sum of series. The symbol Σ means sum. $\sum_{}^{6}(4\mathrm{n}-2)$ means sum of the series for which $T\mathrm{n} = 4\mathrm{n}-2$ for the values from 1 to 6. Thus in expanded form:

$$\sum_{n=1}^{6}(4n-2) = 2+6+10+14+18+22 = 82$$

Unit: Sequence and Series	Topic : The n^{th} term and sum of n terms in Geometric Sequence
Benchmark	**12.3.3.1** Differentiate between the n^{th} term and sum of n terms of the arithmetic and geometric sequences and use of the sigma notation

Learning Objective: By the end of the topic, students will be able to;



- identify patterns, rules and write the n^{th} term expression or formula for sequences.
- use the geometric sequence ideas and formulas to find the n^{th} term and the sum of n terms.
- use sum of n terms of GP formula to solve problems involving summation (sigma)

Essential questions:



- What is a geometric sequence and how does it differ from an arithmetic sequence?
- How is the n^{th} term in a geometric sequence calculated?

Key Concepts(ASK-MT)	
Attitudes/Values	Confidently describe the patterns and evaluating the nth term in geometric sequences.
Skills	Sequencing and evaluating n^{th} terms in geometric sequences
Knowledge	Geometric Sequences and the n^{th} term.
Mathematical Thinking	Think about how to find the n^{th} term and sum of n terms in a geometric sequence.

Content Background

1. Geometric Sequence

The sequence 2, 6, 18, 54, 162 … is a geometric sequence or geometric progression. In a geometric sequence, each term after the first is obtained from the preceding one by multiplying it by a constant number. This constant number is called the common ratio of the sequence. The value of the common ratio is found by dividing any term by the one immediately preceding it. In the geometric sequence above, the common ratio is 3; $\left(\frac{6}{2}=\frac{18}{6}=\frac{54}{18}=\frac{162}{54}\right)$..*etc*.). The common ratio can be obviously be used to find the next term in a geometric sequence. In the sequence above, the next term is 162 x 3 = 486.

2. The nth term of a geometric sequence

Any sequence in geometric progression can be written $a, ar^2, ar^3, ar^4 \ldots$ where a is the first term and r is the common ratio. The general term can be found by examining the following table;

T_1	T_2	T_3	T_4	T_5	…….	T_n
a	ar	ar^2	ar^3	ar^4	…….	ar^{n-1}

The index r in any term is one less than the number of the term. Hence, for the geometric sequence $a, ar^2, ar^3, ar^4 \ldots$ the n^{th} term is given by $T_n = ar^{n-1}$.

3. The sum of n terms of a geometric series

The sum S_n of the first n terms of the geometric series $a, ar, ar^2, ar^3, ar^4 \ldots\ldots ar^{n-1}$ is given by

$$S_n = \frac{a(1-r^n)}{1 \quad r} \; or \; S_n = \frac{a(r^n-1)}{r-1}$$

Example 1

Find the sum of the first 9 terms of the series 32 + 16 + … can be calculated as

$$Sn = \frac{a(1-r^n)}{1-r}$$

$$S_9 = \frac{32(1-{}^1\!/_2{}^9)}{1-{}^1\!/_2}$$

$$= 64\left[1-\left(\frac{1}{2}\right)^9\right]$$

$$= 63.875$$

Example 2

Find $\sum_{n=1}^{8} \square 3 \text{ x } 2^{n-1}$,

The series is 3 + 6 + 12 + …..+ 384.
The series is geometric series with $a = 3, r = 2,$ and $n = 8$ we can use $S_n = \frac{a(r^n-1)}{r-1}$

Thus $\sum_{n=1}^{8} \square 3 \text{ x } 2^{n-1} = S_8 = \frac{3(2^8-1)}{2-1} = 765$

Unit: Sequence and Series	Topic : Properties of Arithmetic and Geometric Sequences and Series
Benchmark	**12.3.3.2** Apply the properties of arithmetic and geometric sequences and series to solve problems.

Learning Objective: By the end of the topic, students will be able to;



- solve problems involving arithmetic and geometric sequences and series, and
- solve problems in real life involving arithmetic and geometric sequences and series.

Essential questions:



- What is the difference between a sequence and a series?
- What are some problems in life that involves the application of either arithmetic or geometric sequences and series?

Key Concepts(ASK-MT)	
Attitudes/Values	Appreciate the application of sequences and series in solving real problems.
Skills	Solving real life problems involving sequences and series.
Knowledge	Application of arithmetic and geometric sequences and series.
Mathematical Thinking	Think about how to apply geometric and arithmetic sequences and series in solving real life problems.

Content Background

An important practical application of geometric sequences is the connection with compound interest, superannuation and time payments.

1. Compound Interest

We can use the knowledge of geometric sequences to develop a general formula for finding the amount A kina that P Kina becomes when invested for n years at r % p.a. compound interest.
K100 amounts to $(100+r)$ in one year.
Therefore, K1 amounts $1+\frac{r}{100}$ in one year

Applying this investment of *P* kina we have P kina amounts to $1+\frac{r}{100}$ in the first year

$P\left(1+\frac{r}{100}\right)$ amounts to $P(1+\frac{r}{100})^2$ in the second year
$P(1+\frac{r}{100})^2$ amounts to $P(1+\frac{r}{100})^3$ in the third year

The successive amounts, in Kina, form a geometric sequence.

$$P(1+\frac{r}{100}), P(1+\frac{r}{100})^2, P\left(1+\frac{r}{100}\right)^3 \ldots\ldots$$

We can see this sequence as geometric with the first term of $P\left(1+\frac{r}{100}\right)$ and a common ratio of $(1+\frac{r}{100})$. The nth term of this geometric sequence is given by $\mathrm{Tn} = ar^{n-1}$

i.e.
$$\mathrm{T_n} = P(1+\frac{r}{100})(1+\frac{r}{100})^{n-1}$$
$$= P(1+\frac{r}{100})^n$$

Thus when P kina is invested for n years at r% p.a. compound interest, it amounts to A where $A = P\left(1+\frac{r}{100}\right)^n$

Example

Find what K800 amounts to in 6 years if it is invested at 8% p.a. compounded yearly.

Solution

$$A = P\ 1 + \frac{r}{100}^{\ n}$$
$$= 800\ 1 + \frac{8}{100}^{\ 6}$$
$$= 800 \text{ x } 1.086$$
$$= 1269.4994$$

Thus after six years, K800 amounts to K1269.50.

In the example above, the interest was compounded annually. That is, at the end of each year the interest is calculated and the principal for the next year is increased by this amount. Interest is often compounded quarterly, monthly or even daily.

In general, the compound interest formula is $A_n = P(1 + \frac{r}{100})^n$

where, r is the rate % per period.
P is the principal and
A_n is the amount after *n* periods

2. Superannuation

A man invests K750 at the beginning of each year in a superannuation scheme. If the interest is paid at the rate of 8% p.a. on the investment (compounded annually), how much will his investment be worth after 20 years?

Solution

The first K750 will be invested for 20 years and will amount to K750 x 1.08^{20}
The second K750 will be invested for 19 years and will amount to K750 x 1.08^{19}
The third K750 will be invested for 18 years and will amount to K750 x 1.08^{18}
And so on until the last K750 which will be invested for 1 year and will amount to K750 x 1.08.
At the end of 20 years his investment will be worth
K750 (1.08^{20} + 1.08^{19} + 1.08^{18} + … + 1.08)
The series in the parentheses can be written
1.08 + 1.08^2 + 1.08^3 + … + 1.08^{20}

This is a geometric series with the first term of 1.08 and a common ratio of 1.08. Thus its sum is

$$S_{20} = \frac{a(r^n - 1)}{r - 1} = \frac{1.08(1.08^{20} - 1)}{1.08 - 1}$$

Therefore the value of the total investment = $\frac{750 \text{ x } 1.08(1.08^{20} - 1)}{0.08} = 3,7067$ (to the nearest kina)

3. Time Payments

A man borrowed K5000 at the rate of 18% p.a. (1.5% per month) where the interest is compounded monthly on the balance owing. The loan is repaid in equal monthly installments. What should the amount of each installment be, to pay off the loan in 4 years?

Solution

Let the monthly installment be *P* Kina and the amount owing after *n* months be Kina. After 1 month and after paying the first installment of *P* kina the amount owing;

$$A_1 = 500 \text{ x } 1.015 - P$$

After 2 months, $A_1 = A_1 \text{ x } 1.015 - P$

$= (500 \text{ x } 1.015\text{-P}) \ 1.05\text{-P}$

$= 5000 \text{ x } 1.015\text{P} - \text{P}$

$= 500 \text{ x } 1.015^2 - \text{P}(1+1.015)$

After 3 months,

$$A_3 = A_2 \text{ x } 1.015 - P$$

$$= 500 \text{ x } 1.015^3 - P(1.015 + 1.0152) - P$$

Continue the pattern: $= 500 \text{ x } 1.015^3 - P(1 + 1.015 + 1.05^2)$

After *n* months, $A_n = 500\,x\,1.015^n - P(1 + 1.015 + 1.015^2 + ... + 1.015^{n-1})$

Therefore after 48 months, $A_{48} = 500 \text{ x } 1.015^{48} - P(1 + 1.015 + 1.015 \ + ... + 1.015^{\ 7})$

But the loan is repaid in full after 48 months, thus $A_{48} = 0$.

Hence $0 = 500\,x 1.015^{48} - P(1 + 1.015 + 1.015^2 + ... + 1.015^{47})$

Therefore, $P = \dfrac{500x\,1.015^{48}}{1 + 1.015 + 1.015^2 + ... + 1.015^{47}}$

Now the denominator is a geometric series in which a = 1, *r* = 1.015 and *n* = 48 thus its sum is $\dfrac{1(1.015^{48} - 1)}{1.015 - 1} = \dfrac{1.015^{48}}{0.015}$

Therefore $P = \dfrac{500x\,1.015^{48}}{1 + 1.015 + 1.015^2 + ...1.015^{47}}$. Each monthly installment must be K146.87

Note: The simple interest rate (r %) which is equivalent to the true interest rate used in the above example is found as follows.

Using $r = \dfrac{100I}{pt}$ where I is the total interest paid
p is the principal
t is the time in years

$$r = \frac{100(146.87\text{x } 48 - 500)}{500 \text{ x } 4} = 10.25$$

Therefore, the equivalent simple interest rate is 10.25% p.a.

Unit: Sequence and Series	Topic : Infinite Geometric Series

Benchmark	**12.3.3.2** Apply the properties of arithmetic and geometric sequences and series to solve problems.

Learning Objective: By the end of the topic, students will be able to;

- use the concept of "limiting sum" or "sum to infinity" correctly,
- connect the idea of sum to infinity to the sum of *n* terms of a GP in terms of limits as *n* gets larger and larger in $S_n = \frac{a(1-r^n)}{1-r}$ when $r < 1$. and
- calculate sum to infinity and solve problems in real life situations.



Essential questions:



- When is there 'limiting sum' and 'sum to infinity'?
- How is the sum to infinity calculated?

Key Concepts(ASK-MT)	
Attitudes/Values	Confidently explain the concepts of limiting sum and sum to infinity in terms of a geometric progression.
Skills	Analyse and evaluate limiting sums and sum to infinity.
Knowledge	Limiting sum, sum to infinity.
Mathematical Thinking	Think about how to calculate the sum to infinity and solve problems in real life situations.

Content Background

Consider the series $\frac{1}{2}2+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\ldots\ldots\ldots\ldots$

The sum of *n* terms $S_n = \frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\ldots\ldots+\left(\frac{1}{2}\right)^n$

Now the sum of two terms $Sn = \frac{3}{4} = 1-\left(\frac{1}{2}\right)^2$

the sum of three terms $S_3 = \frac{7}{8} = 1-\left(\frac{1}{2}\right)^3$

the sum of four terms $S_4 = \frac{7}{8} = 1-\left(\frac{1}{2}\right)^4$

the sum of five terms $S_5 = \frac{7}{8} = 1-\left(\frac{1}{2}\right)^5$

the sum of six terms $S_6 = \frac{7}{8} = 1-\left(\frac{1}{2}\right)^6$

It seems that the more terms we take the closer the sum tends to 1. Now using the formula for the sum of *n* terms of a geometric series we have $a = \frac{1}{2}, r = \frac{1}{2}$. Thus $S_n = \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = 1-\left(\frac{1}{2}\right)^n$

Now as *n* becomes larger and larger, $\left(\frac{1}{2}\right)^n$ becomes smaller and smaller. If *n* is very large $\left(\frac{1}{2}\right)^n \xrightarrow{\square} 0\left[\left(\frac{1}{2}\right)^n tends\ to\ zero\right]$ and $S_n \xrightarrow{\square} 1$.

When the sum of a series tends to a particular value in this way, we say that this value is the limiting sum of the series. In the example above the limiting sum is 1. In general for a geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

Now if $|r|<1 (i.e.\ if -1<r<1)$ then r^n becomes smaller and smaller as *n* becomes larger and larger. We say the limit of rn as n approaches infinity equals zero. In symbols $\lim_{n\to\infty}(r)^n = 0$ for $|r|<1$.Hence the limiting sum is $\frac{a}{1-r}$.

If *S* is the limiting sum of a geometric series with |r|< 1 then; $S = \frac{a}{1-r}$.

Note: A geometric series with |r| > 1 can have no limiting sum since the terms grow larger and larger as n increases.

Example: Find the limiting sum of this geometric series; 6,3,1.5,0.75,,,

Solution: The first term is 6 and the common ratio is 0.5. Substituting into

$$S_\infty = \frac{a}{1-r} = \frac{6}{1-0.5} = \frac{6}{0.5} = 12.$$

Unit: Binomial Theorem	Topic : Pascal's Triangle
Benchmark	**12.3.3.3** Use the binomial theorem and the Pascal's triangle to expand binomial expression.

Learning Objective: By the end of the topic, students will be able to;



- generate Pascal's triangle,
- expand a binomial expression using Pascal's triangle, and
- use the binomial theorem to expand a binomial expression

Essential questions:



- What is a Pascal's triangle?
- How useful is a Pascal's triangle?
- What is the binomial theorem?
- How a binomial expression is expanded using the binomial theorem?

Key Concepts(ASK-MT)	
Attitudes/Values	Recognise the pattern from the Pascal's triangle and expand the triangle.
Skills	Expand the Pascal's triangle to identify the pattern.
Knowledge	Pascal's triangle and binomial theorem.
Mathematical Thinking	Think about how to expand a binomial expression using the Pascal's triangle and also the binomial theorem.

Content Background

1. Binomial Expression

A binomial expression is the sum, or difference, of two terms. *For example,*

$$x+1, 3x+2y, a-b$$

are all binomial expressions. You will be familiar already with the need to expand brackets when squaring such quantities. You will know, for example, that

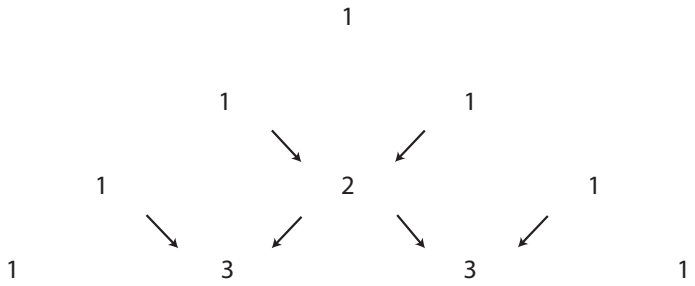
$$\begin{aligned}(x+1)^2 &= (x+1)(x+1)\\ &= x^2+x+x+1\\ &= x^2+2x+1\end{aligned}$$

If we want to raise a binomial expression to a power higher than 2 (for example if we want to find $(x+1)^7$ it is very cumbersome to do this by repeatedly multiplying $x+1$ by itself. Hence you will learn how a triangular pattern of numbers, known as Pascal's triangle, can be used to obtain the required result very quickly.

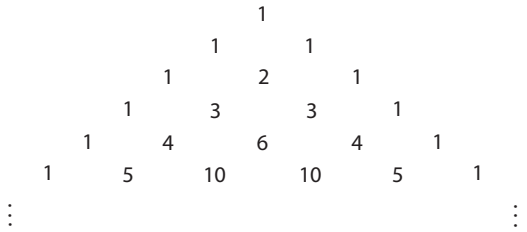
2. Pascals Triangle

We start to generate Pascal's triangle by writing down the number 1. Then we write a new row with the number 1 twice. We then generate new rows to build a triangle of numbers. Each new row must begin and end with a 1.

The remaining numbers in each row are calculated by adding together the two numbers in the row above which lie above-left and above-right. So, adding the two 1s in the second row gives 2, and this number goes in the vacant space in the third row. The two vacant spaces in the fourth row are each found by adding together the two numbers in the third row which lie above-left and above-right: 1 + 2 = 3, and 2 + 1 = 3. This gives:



We can continue to build up the triangle in this way to write down as many rows as we wish. The first six rows of Pascal's triangle is given for you as shown below.



If we want to expand $(a+b)^3$ we select the coefficients from the row of the triangle beginning 1,3: these are 1,3,3,1. We can immediately write down the expansion by remembering that for each new term we decrease the power of *a*, this time starting with 3, and increase the power of *b*. So $(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$ which we would normally write as just $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Thinking of $(a+b)^3$ as

$$(a+b)(a^2+2ab+b^2) = a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

we note that the term $3ab^2$, for example, arises from the two terms ab^2 and $2ab^2$; again this is the link with the way 3 is generated in Pascal's triangle - by adding the 1 and 2 in the previous row.

Example

Suppose we wish to find $(a+b)^4$. To find this we use the row beginning 1,4, and can immediately write down the expansion.

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

We can apply the same procedure to expand any binomial expression, even when the quantities *a* and *b* are more complicated. Consider the following examples.

Example

Suppose we want to expand $(2x+y)^3$.

We pick the coefficients in the expansion from the relevant row of Pascal's triangle: (1,3,3,1). As we move through the terms in the expansion from left to right we remember to decrease the power of $2x$ and increase the power of y. So,

$$\begin{aligned}(2x+y)^3 &= 1(2x)^3 + 3(2x)^2y + 3(2x)^1y^2 + 1y^3 \\ &= 8x^3 + 12x^2y + 6xy^2 + y^3\end{aligned}$$

Unit: Binomial Theorem	Topic : Factorial, Permutation and Combination
Benchmark	**12.3.3.3** Use the binomial theorem and the Pascal's triangle to expand binomial expression.

Learning Objective: By the end of the topic, students will be able to;

- calculate factorials of numbers,
- define and use permutation and combination,
- distinguish the difference between permutation and combination,
- appreciate the relationship between combination and Pascal's Triangle, and
- solve problems using permutation and combination.



Essential questions:

- What are factorials?
- When are the concepts of permutation and combination applied?
- What is the difference between permutation and combination?



Key Concepts(ASK-MT)	
Attitudes/Values	Appreciate the application of permutation and combination to solve real life problems.
Skills	Solving real life problems using permutation and combination.
Knowledge	Factorials, Permutation, Combination.
Mathematical Thinking	Think about how to calculate factorials and use permutation or combination to solve problems.

Content Background

1. Factorial

If n is a positive integer, then the factorial of n denoted by $n!$ and defined as the product of n +ve integers from n to 1 (or 1 to n)

i.e., $n!$ = $n(n\text{-}1)(n\text{-}2)$……3.2.1

For example

4! = 4.3.2.1 = 24

And 6! = 6.5.4.3.2.1 = 720

One important relationship concerning factorials is that

$(n+1)! = (n+1)n!$ ---------- $(1)!$ _____ (1)

For instance,

5! = 5.4.3.2.1

= 5(5.4.3.2.1)

5! = 5.4!

Obviously, 1! = 1 and this permit to define from equation (1)

$$n! = \frac{(n+1)!}{n+1}$$

Substitute 0 for n, we obtain

$$0! = \frac{(0+1)!}{0+1} = \frac{1!}{1} = \frac{1}{1}$$

$$0! = 1$$

$$0! = 1$$

2. Combination

Each of the groups or selections which can be made out of a given number of things by taking some or all of them at a time is called **combination**.

In combination the order in which things occur is not considered *e.g.*; combination of a,b,c taken two at a time are ab,bc,ca.

The numbers $\binom{n}{r}$ *or* ${}^{n}C_{r}$ and is defined as $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

e.g., $\binom{6}{4} = \frac{6!}{4!(6-4)} = \frac{6 \text{ x } 5 \text{ x } 4}{4! \text{ x } 2!} = \frac{6 \text{ x } 5}{2 \text{ x } 1} = 15$

Example 1: Expand $\binom{7}{3}$

Solution:

$$\binom{7}{3} = \frac{7!}{3!(7-3)!}$$
$$= \frac{7.6.5.4!}{3.2.1.4}$$
$$= 35$$

This can also be expand as

$$\binom{7}{3} = \frac{7.6.5}{3.2.1} = 35$$

If we want to expand $\binom{7}{5}$, then $\binom{7}{5} = \left(\frac{7.6.5.4.3}{5.4.3.2.1}\right) = 21$

Procedure: Expand the above number as the lower number and the lower number expand till 1.

Some Important Results

(i) $\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1xn!} = 1$

(ii) $\binom{n}{n} = \frac{n!}{n!(n-n)} = \frac{n!}{n!0!} = \frac{n!}{n! \text{ x } 1} = 1$

(iii) $\binom{n}{r} = \frac{n}{r\,(n-r)}$

For example

$\binom{4}{0} = \binom{4}{4} = 1$ as $\frac{4!}{0!(4-0)!} = \frac{4!}{4!0!} = 1$

Note: The numbers $\binom{n}{r}$ or ${}^{n}C_{r}$ are also called **binomial co-efficients.**

3. Permutation

Define **permutation** as an **ordered arrangement** of a specific number of objects selected from a set. The number of distinct permutations of *r* objects taken from n objects is given by

$$nPr = \frac{n!}{(n-r)!}$$

Example: How many different arrangements are possible if 3 people are to occupy 3 executive positions of president, treasurer and secretary?

Solution: Person 1 can be president, treasurer or secretary. The same applies to the other two. One such arrangement is; Person is President, Person 2 is Treasurer and Person 3 is secretary. Another is Person 1 remains President and Person 2 and Person 3 change positions. Here position is important or we say ORDER is important. Such different arrangements taking into account who occupies which position is permutation. Here we pick 3 people from 3 so we have

$$3P3 = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3!}{1} = 3! = 3 \text{ x } 2 \text{ x } 1 = 6$$

Note: Difference between Permutation and Combination is ORDER. If order is involved its permutation.

Unit: Binomial Theorem	Topic : Binomial Theorem
Benchmark	**12.3.3.3** Use the binomial theorem and the Pascal's triangle to expand binomial expression.

Learning Objective: By the end of the topic, students will be able to;



- discover the properties of binomial theorem expansion
- apply binomial theorem to expand and calculate specific terms of binomials.
- solve problems involving binomial theorem

Essential questions:



- What are the properties of the binomial theorem?
- How are the binomial expressions expanded?

Key Concepts(ASK-MT)	
Attitudes/Values	Be content that binomial theorem is a short and precise method of expanding binomials.
Skills	Apply binomial theorem to expand a binomial expression.
Knowledge	Binomial theorem and its properties.
Mathematical Thinking	Think about how to expand binomials and calculate specific terms in the binomial expression.

Content Background

1. Binomial Theorem

If we wanted to expand a binomial expression with a large power, e.g. $(1+x)^{32}$, use of Pascal's triangle would not be recommended because of the need to generate a large number of rows of the triangle. An alternative method is to use the binomial theorem. The theorem enables us to expand $(a+b)^n$ in increasing powers of b and decreasing powers of a. We will look at expanding expressions of the form $(a+b)^2,(a+b)^3...,(a+b)^{32},..,$ that is when the power is a positive whole number. Under certain conditions the theorem can be used when n is negative or fractional and this is useful in more advanced applications, but these conditions will not be studied here.

When n is a positive whole number,

$$(a+b)^n=\binom{n}{0}a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}+\binom{n}{3}a^{n-3}b^3.....+\binom{n}{r}a^{n-r}b^r.....+\binom{n}{n}b^n.....(1)$$

Or briefly, $(a+b)^n=\sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$

Remarks: The coefficients of the successive terms are $\binom{n}{0},\binom{n}{1},\binom{n}{2},---,\binom{n}{r}---,\binom{n}{n}$ and are called *Binomial Coefficients.*

Note: Sum of binomial coefficient is 2^n.

2. Another form of the Binomial Theorem:

$$(a+b)^n=a^n+\frac{n}{1!}a^{n-1}b+\frac{n(n-1)}{2!}a^{n-2}b^2+\frac{n(n-1)(n-2)}{3!}a^{n-3}b^3+.....$$
$$+\frac{n(n-1)(n-2)---(n-r+1)}{r!}a^{n-r}b^r+.....+b^n.....(2)$$

The following points can be observed in the expansion of $(a+b)^n$.

1. There are $(n+1)$ terms in the expansion.
2. The 1st term is a^n and $(n+1)^{th}$ term or the last term is b^n.
3. The exponent of 'a' decreases from n to zero.
4. The exponent of 'b' increases from zero to n.
5. The sum of the exponents of a and b in any term is equal to index n.
6. The co-efficients of the term equidistant from the beginning and end of the expansion are equal as $\binom{n}{r}=\binom{n}{n-r}$.

3. The General Term

The term $\binom{n}{r}a^{n-r}b^r$ in the expansion of binomial theorem is called the General term or $(r+1)^{th}$ term. It is denoted by T_{r+1}. Hence $T_{r+1}=\binom{n}{r}a^{n-r}b^r$.

Note: The General term is used to find out the specified term or the required co-efficient of the term in the binomial expansion.

Example: Expand $(x+y)^4$ by binomial theorem.

Solution

$$\begin{aligned}(x+y)^4 &= x^4+\binom{4}{1}x^{4-1}y+\binom{4}{2}x^{4-2}y^2+\binom{4}{3}x^{4-3}y^3+y^4\\ &= x^4+4x^3y+\frac{4\text{ x }3}{2\text{ x }1}x^2y^2+\frac{4\text{ x }3\text{ x }2}{3x2x1}xy^3+y^4\\ &= x^4+4x^3y+6x^2y^2+4xy^3+4\end{aligned}$$

Example: Expand by binomial theorem $\left(a-\frac{1}{a}\right)^6$

Solution

$$\left(a-\frac{1}{a}\right)^6=a^6+\binom{6}{1}a^{6-1}\left(-\frac{1}{a}\right)^1+\binom{6}{2}a^{6-2}\left(-\frac{1}{a}\right)^2+\binom{6}{3}a^{6-3}\left(-\frac{1}{a}\right)^3+\binom{6}{4}a^{6-4}\left(-\frac{1}{a}\right)^4+\binom{6}{5}a^{6-5}\left(-\frac{1}{a}\right)+\binom{6}{6}a^{6-6}\left(-\frac{1}{a}\right)^6$$

4. The Nth Term

$$(x+y)^n=\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$$

If you are asked to find the 10th term of an expansion, substitute *k* = 9 in the first formula and substitute *k* = 10 in the second formula depending on which one you want to use.

Example: Find the 8th term in the expansion of $(x-3)^{15}$.

Solution: We will use both formulas above.

$$(x+y)^n=\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$$

In the above, *n* = 15, 8th term so *k* will be 1 less so it will be 7. We substitute into the formula above

$$\begin{aligned}(x-3)^{15}=(x+(-3))^{15}&=\binom{15}{7}(x)^{15-7}(-3)^7=\binom{15}{7}(x)^8(-2187)\\ &=(6435)x^8(-2187)=-14073345x^8\end{aligned}$$

Using the second formula,just substitute k = 8

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{k-1} x^{n-k+1} y^{k-1}$$

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{k-1} x^{n-k+1} a^{k-1} = \binom{15}{8-1} (x)^{15-8+1} (-3)^{8-1}$$

$$= \binom{15}{7} x^8 (-3)^7$$

$$= -14073345x^8$$

Unit: Determinant	Topic : Determinants
Benchmark	**12.3.3.4** Solve systems of linear equation in two or three variables using determinants.

Learning Objective: By the end of the topic, students will be able to;



- define what a determinant is,
- identify and evaluate linear equations in two variables using determinants, and
- identify and evaluate linear equations in three variables using determinants.

Essential questions:



- What are some examples of linear equations with two variables?
- What are some examples of linear equations with three variables?
- What is a determinant?
- How are determinants represented?

Key Concepts(ASK-MT)	
Attitudes/Values	Confidently solve linear equations with two and three variable using the concept of determinants.
Skills	Apply the concept of determinants to solve system of linear equations.
Knowledge	Determinants and linear equations with two and three variables.
Mathematical Thinking	Think about how to find determinants and solve linear equations through the concept of determinants.

Content Background

1. What is a Determinant?

A quantity obtained by the addition of products of the elements of a square matrix according to a given rule.

A square array of numbers or variables enclosed between vertical lines is called a *determinant*. A determinant is different from a matrix in that a determinant has a numerical value, whereas a matrix does not. The following determinant has two rows and two columns, three rows and three columns.

For instance:

Two rows by two columns: $\mathrm{A}=\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

Three rows by three columns: $\mathrm{A}=\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 3 & 2 \end{bmatrix}$

2. Calculating Determinants

A system of linear equations like;

$$\begin{aligned} a_1x+b_1y&=c_1 \\ a_2x+b_2y&=c_2 \end{aligned}$$ can be represented as $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

This system of equations has a unique solution or not, is determined by the value

3. Determinant of a Two by Two system

Key concept: If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A is written as $|A|=\begin{bmatrix} a & b \\ c & d \end{bmatrix}=(ad-cb)=\det(A)$

Example 1

Evaluate $A=\begin{bmatrix}2 & 4\\ -1 & 2\end{bmatrix}$

Solution: We have $\det(A)=|A|=\begin{bmatrix}2 & 4\\ -1 & 2\end{bmatrix}=2(2)-4(-1)=4+4=8$

So $\det(A)=8$

4. Determinant of a three by three system

Key concept: If $A=\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}$, then determinant of A is written as;

$$|A|=\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}=a\begin{bmatrix}e & f\\ h & i\end{bmatrix}-b\begin{bmatrix}d & f\\ g & i\end{bmatrix}+c\begin{bmatrix}d & e\\ g & h\end{bmatrix}=\det(A)$$

Example 2

Evaluate $A=\begin{bmatrix}2 & 7 & -3\\ -1 & 5 & -4\\ 6 & 9 & 0\end{bmatrix}$

Solution: We have $\det(A)=|A|=\begin{bmatrix}2 & 7 & -3\\ -1 & 5 & -4\\ 6 & 9 & 0\end{bmatrix}=2\begin{bmatrix}5 & -4\\ 9 & 0\end{bmatrix}-7\begin{bmatrix}-1 & -4\\ 6 & 0\end{bmatrix}+(-3)\begin{bmatrix}-1 & 5\\ 6 & 9\end{bmatrix}$

$$\begin{aligned}\det(A)=|A| &= 2\begin{bmatrix}5 & -4\\ 9 & 0\end{bmatrix}-7\begin{bmatrix}-1 & -4\\ 6 & 0\end{bmatrix}+(-3)\begin{bmatrix}-1 & 5\\ 6 & 9\end{bmatrix}\\ &=((0)-(-36))-7((0)-(-24))+(-3)((-9)-(30))\\ &=2(36)-7(24)-3(-39)\\ &=72-168+117\\ \det(A) &= 21\end{aligned}$$

Unit: Limits	Topic : Finding Limits numerically and graphically
Benchmark	**12.3.3.5** Recognize limits from graphs and tables.

Learning Objective: By the end of the topic, students will be able to;

- recognise and calculate the limits from graphs and tables,
- Calculate limits of functions from their graphs.



Essential questions:

- What is a 'limit'?
- Why are there limits to functions?



Key Concepts(ASK-MT)	
Attitudes/Values	Be assertive and confident to explain and examine the values of functions as *x* approaches specific limits.
Skills	Calculate limits of functions.
Knowledge	Limits of functions.
Mathematical Thinking	Think about how to examine the limits to functions from graphs and tables.

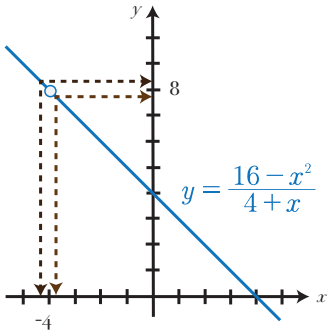
Content Background

1. Limits of a Function – An Informal Approach

The two broad areas of calculus known as differentiation and integral calculus are built on the foundation of a Limit. Consider the function $f(x)=\frac{16-x^2}{4+x}$ whose domain is the set of all real numbers except - 4.

x	-4.1	-4.01	-4.001
$f(x)$	8.1	8.01	8.001

x	-3.9	-3.99	-3.999
$f(x)$	7.9	7.99	7.999



Although f cannot be evaluated at - 4 because substituting - 4 for x results in the quantity *0/0, f(x)* can be calculated at any number x that is very close to -4. The two tables show that as x approaches - 4 from either the left or right, the function values $f(x)$ appears to be approaching 8, in other words, when x is near - 4, $f(x)$ is near 8. To interpret the numerical information graphically, observe that for every number $x \neq$ - 4, the function f can be simplified by cancellation:

$$f(x)=\frac{16-x^2}{4+x}=\frac{(4+x)(4-x)}{4+x}=4-x \qquad \text{when } x \text{ is near -4 } f(x) \text{ is near 8}$$

As shown in the figure on the previous page, the graph of f is essentially the graph of $y=4-x$ with the exception that the graph of f has a hole at the point that corresponds to $x=-4$. For x sufficiently close to -4, represented by two arrowheads on the $x-\text{axis}$, the two arrowheads on the $y-\text{axis}$, representing the function values $f(x)$, simultaneously get closer and closer to the number 8. We say 8 is the limit of $f(x)$ as x approaches -4.

The discussion of the limit concept is facilitated by a special notation. If we let the arrow symbol $\rightarrow$ represent the word approach, then the symbolism

$x \rightarrow a$ signifies that x approaches a from both sides. In other words, from the left and right sides of a on a number line. In the left- hand side table above we are letting $x \rightarrow -4^-$ (for example, -4.0001is to the left of the number line), whereas in the right-hand side table $x \rightarrow -4^+$

Unit: Limits	Topic : Finding Limits algebraically using the limit laws
Benchmark	**12.3.3.6** Find limits of sums, differences, products, quotients and rational functions.

Learning Objective: By the end of the topic, students will be able to;

- calculate limits using fundamental limit theorems.



Essential questions:

- How are limits calculated in terms of sum, differences, products and quotients rules?
- How can limits of rational expressions be calculated?



Key Concepts(ASK-MT)	
Attitudes/Values	Be confident and evaluate limits to rational expressions with sum, differences, products and quotients.
Skills	Calculate limit using fundamental limit theorems.
Knowledge	Limits of functions with sum, differences, products and quotients.
Mathematical Thinking	Think about how to evaluate limits to functions either by sum, differences, products or quotients.

Content Background

1. Fundamental Limit Theorems

(i) $\lim_{x\to a} c = c$, where c is a constant
(ii) $\lim_{x\to a} x = a,$,

Example

(a) Using theorem (i), $\lim_{x\to 2} 10 = 10$ and $\lim_{x\to 6} \pi = \pi$
(b) Using theorem (ii), $\lim_{x\to x} x = 2$ and $\lim_{x\to 0} x = 0$

2. Limit of a constant Multiple

The limit of a constant multiple of a function f is the constant times the limit of f as x approaches a number a. Thus if c is a constant, then $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$.

Example

(a) $\lim_{x\to 8} 5x = 5 \lim_{x\to 8} x = 5.8 = 40$

(b) $\lim_{x\to -2}\left(-\frac{3}{2}x\right) = -\frac{3}{2}\lim_{x\to -2} x = -\frac{3}{2}.(-2) = 3.$

3. Limit of a Sum, Product, Quotient and Power

Suppose a is a real number and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$, then

(i) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L_1 \pm L_2,$

(ii) $\lim_{x\to a}[f(x)g(x)] = (\lim_{x\to a} f(x))(\lim_{x\to} (g(x)) = L_1 L_2,$ and

(iii) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L_1}{L_2}, L_2 \neq 0.$

(iv) $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^2 = L^n.$

Example

(a) $\lim_{x\to 5}(10x+7)=\lim_{x\to 5}(10x)+\lim_{x\to 5}(x)+\lim_{x\to 5}7=10.5+7=57$

(b) $\lim_{x\to 5}\frac{5}{x^2}=\frac{\lim_{x\to 5}5}{\lim_{x\to 5}x^2}=\frac{5}{5^2}=\frac{5}{25}=\frac{1}{5}$

(c) $\lim_{x\to 3}(x^2-5x+6)=\lim_{x\to 3}x^2-\lim_{x\to 3}(5x)+\lim_{x\to 3}6=3^2-5.3+6=0$

4. Limit of a Rational Function

If $p(x)$ and $q(x)$ are two polynomials such that $f(x)=\frac{p(x)}{q(x)}$ then, $\lim_{x\to a}f(x)=\frac{\lim_{x\to a}p(x)}{\lim_{x\to a}q(x)}$ provided $\lim_{x\to a}q(x)\neq 0$.

Example: $\lim_{x\to 1}\frac{x^2+3x+2}{x+1}=\frac{\lim_{x\to 1}x^2+3x+2}{\lim_{x\to 1}x+1}=\frac{6}{2}=3.$ Alternatively,

$$\lim_{x\to 1}\frac{x^2+3x+2}{x+1}=\lim_{x\to 1}\frac{(x+2)(x+1)}{(x+1)}=\lim_{x\to 1}(x+2)=1+2=3.$$

Unit: Limits	Topic : Continuity and Limits
Benchmark	**12.3.3.7** Define continuity in terms of limits and functions.

Learning Objective: By the end of the topic, students will be able to;

- define continuity in terms of limits and functions.
- explain the concept of discontinuity of a function in terms of limits.



Essential questions:

- What is a continuous function?
- How are continuous functions and limits related?

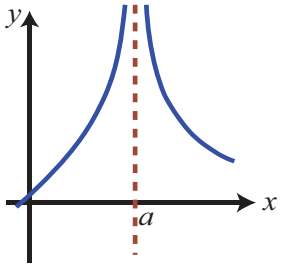


Key Concepts(ASK-MT)	
Attitudes/Values	Be open minded and explain limits to continuous functions.
Skills	Describe limits of continuous functions.
Knowledge	Limits and continuous functions.
Mathematical Thinking	Think about how to calculate limits to continuous functions

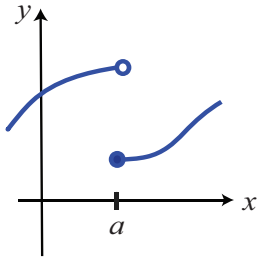
Content Background

1. Continuity

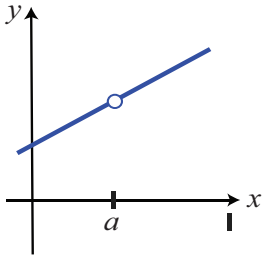
A function is continuous if there are no gaps in the graph of that function and if the limit exists at that point. Illustrated below are some intuitive examples of graphs of functions that are not continues at



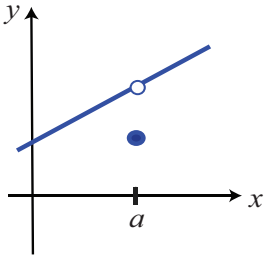
(a) $\lim_{x\to a} f(x)$ *does not exist and* $f(a)$ *is not defined*



(b) $\lim_{x\to a} f(x)$ does not exist and $f(a)$ is defined



(c) $\lim_{x\to a} f(x)$ does not but $f(a)$ is not defined



(d) $\lim_{x\to a} f(x)$ exists, $f(a)$ is defined, *but* $\lim_{x\to a} f(x) \neq f(a)$

2. Definition of a Continuous Function at a number

A function f is said to be continuous at a number a if

(i) $f(a)$ is defined (ii) $\lim_{x\to a} f(x)$ exist, and (iii) $\lim_{x\to a} f(x) = f(a)$

If any one of the three conditions in definition above fails, then f is said to be discontinuous at the number a.

Example: Determine whether each of the functions is continuous at 1.

a) $f(x) = \frac{x^3-1}{x-1}$

(b) $g(x) = \begin{cases} \frac{x^3-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

(c) $h(x) = \begin{cases} \frac{x^3-1}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$

Solutions

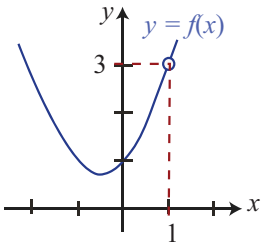
(a) f is continuous at 1 since substituting $x = 1$ into the function results in 0/0. We say that $f(1)$ is not defined and so the first condition of continuity in the definition given above is violated.

(b) Because g is defined at 1, that is, $g(1) = 2$, we next determine whether $\lim_{x\to 1} g(x)$ exists.

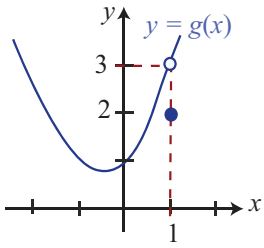
From $\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\to 1}(x^2 + x + 1) = 3$. We conclude $\lim_{x\to 1} g(x)$ exists and equals 3. Since this value is not the same as $g(1) = 2$, the second condition of the definition given above is violated. The function g is discontinuous at 1.

(c) First, $h(1)$ is defined, in this case, $h(1) = 3$. Second, $\lim_{x\to 1} h(x) = 3$. Third, we have $\lim_{x\to 1} h(x) = h(1) = 3$. Thus all the three conditions in definition are satisfied and so the function h is continuous at 1.

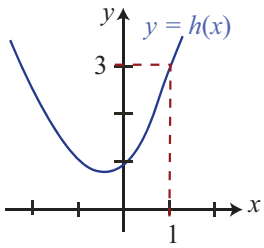
The graphs of the three functions are compared as shown below.



(a)



(b)



(c)

Unit: Differentiation	Topic: The Gradient Function
Benchmark	**12.3.3.8** Calculate the derivatives of functions, including polynomial, rational, trigonometric, logarithmic, inverse, composite and exponential functions.

Learning Objective: By the end of the topic, students will be able to;

- calculate gradient of a function using the first principle formula.



Essential questions:

- What is the purpose of differentiation?
- What is a secant line to a curve?
- How do you explain the relationship of a secant line to a tangent line to a curve?
- How is derivative evaluated using the first principle?



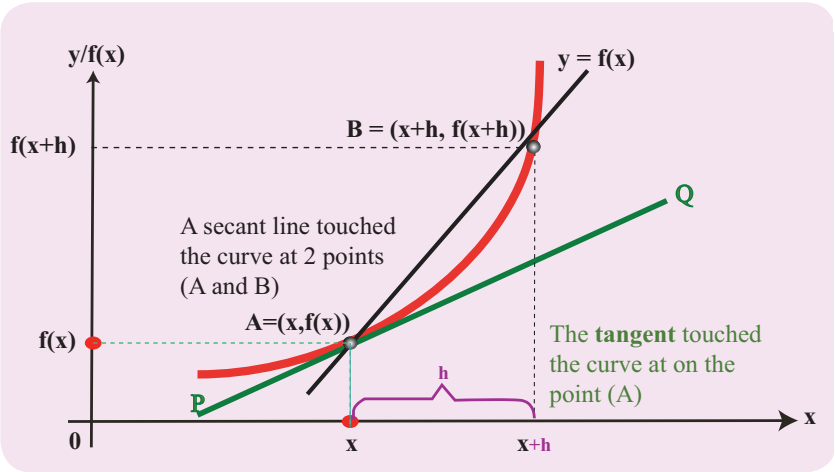
Key Concepts(ASK-MT)	
Attitudes/Values	Appreciate the first principle formula in differentiation.
Skills	Calculate the gradient of a tangent using the first principle of differentiation.
Knowledge	Gradient Function and the First Principle in differentiation.
Mathematical Thinking	Think about how to differentiate functions to obtain gradient functions using the first principle.

Content Background

1. The Gradient Function

The gradient of a straight line (linear function) is ALWAYS CONSTANT. This is not true for functions of degrees greater than 1 such as quadratics, cubes etc. The graphs of such functions are curves and the gradient at various points along the curve is different. Let's consider the illustration on the diagram below.

From the diagram:



- *PQ is the **tangent** for the function f at the point A (green dot)*
- *AB is a **secant** line that touched the function f at the point A and B (grey line).*

As point B moves towards A, the secant line gets to become the tangent at point A. No matter how close the secant moves, it will never become the tangent. This means that the tangent is the limit. We proceed to the basic formula to calculate gradient of the secant; that is rise/run. From the diagram we see that

As point B moves towards A, the secant line gets to become the tangent at point A. No matter how close the secant moves, it will never become the tangent. This means that the tangent is the limit.

We proceed to the basic formula to calculate gradient of the secant; that is rise/run. From the diagram we see that

$$\text{gradient} = \frac{\text{rise}}{\text{run}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$= \frac{f(x+h)-f(x)}{h}.$$

As h or the increment gets smaller and smaller or approaches 0, the secant tends towards to tangent. So the above can be written as $\text{gradient} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. This is the gradient function and the notation for this is $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

This is known as the derivative formula and usually called **First Principle.**

Unit: Differentiation | **Topic : Definition of Derivatives from First Principle and Rules of Differentiation**

Benchmark	**12.3.3.8** Calculate the derivatives of functions, including polynomial, rational, trigonometric, logarithmic, inverse, composite and exponential functions.

Learning Objective: By the end of the topic, students will be able to;

- find the derivative of simple functions from First Principle, and
- use definition of derivative and find the gradient of function at any point.



Essential questions:

- How is the derivative defined from the first principle?
- What are the rules of differentiation?



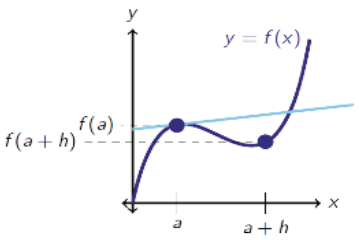
Key Concepts(ASK-MT)	
Attitudes/Values	Appreciate the definition of the first derivative to find the gradient of a function.
Skills	Use definition of a derivative to find gradient.
Knowledge	Definition of derivatives from the first principle.
Mathematical Thinking	Think about how to evaluate derivatives from the first principle

Content Background

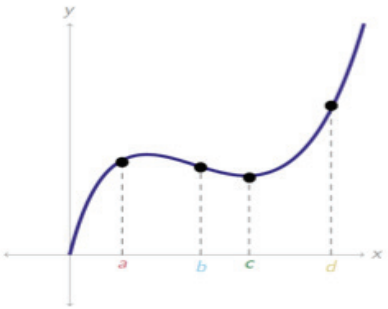
1. Secant and Tangent Line

Recall, the slope of a line is given by any of the following: $\frac{\text{Rise}}{\text{Run}}$ $\frac{\triangle y}{\triangle x}$ $\frac{y_2-y_1}{x_2-x_1}$

Suppose we're interested in the slope of the secant line to $f(x)$ through the points where $x=a$ and $x=a+h$



Slope of secant line: $\frac{f(a+h)-f(a)}{h}$



Slope of tangent line: $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

2. Derivative at a Point

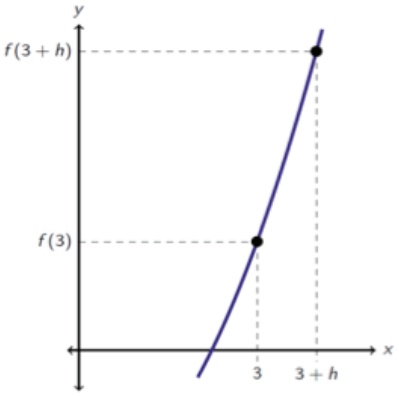
Definition: Given a function $f(x)$ and a point a, the slope of the tangent line to $f(x)$ at *a* is the derivative of *f* at *a*, written $f'(a)$.

So, $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$
$f'(a)$ is also the instantaneous rate of change of f at a.

If $f'(a)>0$, then f is increasing at a. It points "up".
If $f'(a)<0$, then f is decreasing at a. It points "down".
If $f'(a)=0$, then f looks constant at a. It looks "flat"

Use the definition of the derivative to find the slope of the tangent line to $f(x)=x^2-5$ at the point $x=3$.

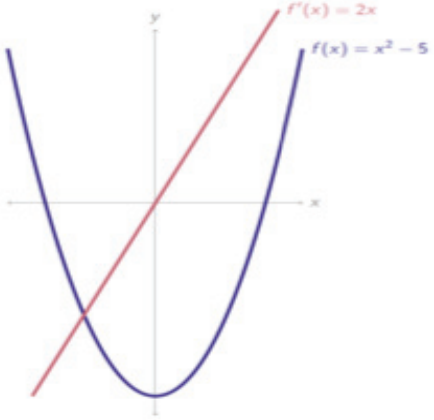
$$\begin{aligned}
f'(3) &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h}\\
&= \lim_{h\to 0}\frac{f((3+h)^2-5)-(3^2-5)}{h}\\
&= \lim_{h\to 0}\frac{(9+6h+h^2-5)-4}{h}\\
&= \lim_{h\to 0}\frac{h^2+6h}{h}\\
&= \lim_{h\to 0} h+6\\
&= 6
\end{aligned}$$



3. The Derivative as a Function

Let's keep the function $f(x)=x^2-5$. We just showed $f'(3)=6$. We can also find its derivative at an arbitrary point x:

$$\begin{aligned}
f'(x) &= \lim_{h\to o}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to o}\frac{f(x+h)^2-5-(x^2-5)}{h}\\
&= \lim_{h\to o}\frac{x^2+2xh+h^2-5-x^2+5}{h}\\
&= \lim_{h\to o}\frac{2xh+h^2}{h}\\
&= \lim_{h\to o} 2x+h\\
&= 2x
\end{aligned}$$



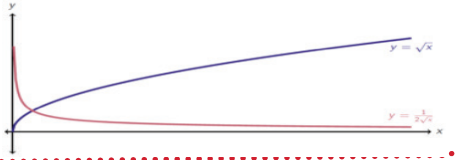
In particular, $f'(3)=6$.

4. Using the Definition of a Derivative

Example: (a) Find the derivative of $f(x)=x^2-x$ from First Principle.

Solution:

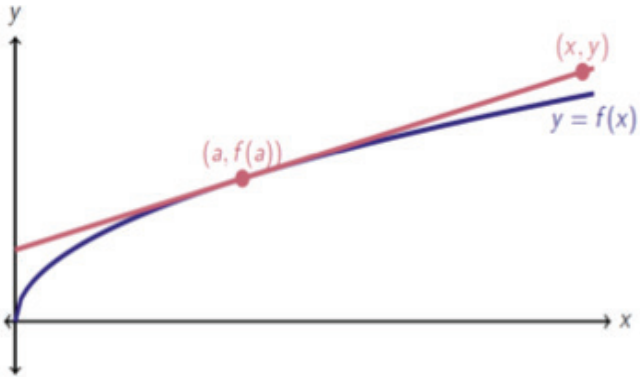
$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-(x+h)-(x^2-x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-(x+h)-(x^2-x)}{h}\\
&= \lim_{h\to 0}\frac{x^2+2hx+h^2-x-h-x^2+x}{h} = \lim_{h\to 0}\frac{2hx+h^2-h}{h}\\
&= \lim_{h\to 0}\frac{h(2x+h-1)}{h} = \lim_{h\to 0} 2x+h-1 = 2x-1\\
f'(x) &= 2x-1
\end{aligned}$$



(b) Let $f'(x)=\sqrt{x}$. Using the definition of a derivative, calculate $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) \\ &= \lim_{h\to 0}\frac{(x+h)-(x)}{(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

5. Equation of a Tangent Line



The tangent line to $f(x)$ at a has slope $f'(a)$ and passes through the point $(a,f(a))$.

Slope of the tangent line: $f'(a)=\frac{rise}{run}=\frac{y-f(a)}{x-a}$

Rearranging: $y-f(a)=f'(a)(x-a)$

….. equation of tangent line

The tangent line to the function $f(x)$ at point a is: $(y-f(a))=f'(a)(x-a)$. In general, a line with slope m passing through point (x_1,y_1) has the equation: $(y-y_1)=m(x-x_1)$

Example 1. Find the equation of the tangent line to the curve $f(x)=\sqrt{x}$ at $x=9$

Recall $\frac{d}{dx}[\sqrt{x}]=\frac{1}{2\sqrt{x}}$. Thus at $a=9$, $f(a)=3$ and $f'(a)=\frac{1}{2\sqrt{x}}=\frac{1}{2(3)}=\frac{1}{6}$

Therefore, $(y-3)=\frac{1}{6}(x-9)y-3=\frac{1}{6}(x-9)\Rightarrow 6y-18=x-9\Rightarrow x-6y+9=0$.

6. Gradient of $y=f(x)$ at any point on the curve.

Example 2: Using the definition of a derivative, find the derivative of this function, $y=3x^2-2x+1$ and the gradient of the curve where $x=4$ and $x=-4$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\left(\frac{[3(x+h)^2-2(x+h)+1]-(3x^2-2x+1)}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{3(x^2+2xh+h^2)-2(x+h)+1-(3x^2-2x+1)}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{6xh+3h^2-2h}{h}\right)=\lim_{h\to 0}\left(\frac{h(6x+3h-2)}{h}\right) \\ &= \lim_{h\to 0}(6x+3h-2)=6x+3(0)-2=6x-2 \end{aligned}$$

Thus at, $x=4$, $f'(4)=6(4)-2=22$ and at $x=-1, f'(-1)=6(-1)-2=-8$,

Unit: Differentiation	Topic: Standard Rules of Differentiation
Benchmark	**12.3.3.8** Calculate the derivatives of functions, including polynomial, rational, trigonometric, logarithmic, inverse, composite and exponential functions.

Learning Objective: By the end of the topic, students will be able to;

- apply the standard rules for differentiation to evaluate derivatives, and
- recognise the simplicity in differentiating using the rules other than using the first principle.



Essential questions:

- What are the standard rules for differentiation?
- What is the difference between using the standard rules of differentiation and using the first principle?



Key Concepts(ASK-MT)	
Attitudes/Values	Appreciate and use the standard rules for differentiation.
Skills	Calculate gradient using standard rules for differentiation.
Knowledge	Standard rules for differentiation.
Mathematical Thinking	Think about how to differentiate using the standard rules of differentiation.

Content Background

STARDARD RULES FOR DIFFERENTIATION

Differentiation from first principle is necessarily quite tedious but there are standard results which can be used to find the derivatives more directly.

i. The derivative of a constant is zero.

If $y = c$, a constant number, then its graph is straight line parallel to the $x-axis$ with gradient zero.

Thus if $y = c, \frac{dy}{dx} = 0$

Example (i): if $y = 7, \frac{dy}{dx} = 0$

ii. The derivative of $y = mx$ is m

We know that the graph of $y = mx$ is a straight line with gradient m.
Thus if $y = -3x, \frac{dy}{dx} = -3$

Example (ii) If $y = -3x, \frac{dy}{dx} = -3$.

(iii) The derivative of x^n is nx^{n-1}

We have seen that, if $y = x^2, \frac{dy}{dx} = 2x$ and if $y = x^3, \frac{dy}{dx} = 3x^2$

In general for rational n, if $y = x^n, \frac{dy}{dx} = nx^{n-1}$

Example (iii) If $y = x^5, \frac{dy}{dx} = 5x^4$

iv. Constant Multipliers

If a is a constant multiplier such that $y = x^5, \frac{dy}{dx} = 5x^4$.

In a function notation the derivative of $af(x)$ is $af'(x)$.

Example (iv): if $y = 4x^3, \frac{dy}{dx} = 4 \text{ x } 3x^2 = 12x^2$

v. Sums and Differences

The derivative of the sum or difference of two or more functions is the sum or difference of their derivatives.

Example (v): if $y = x^3 - 5x^2 + 9x + 8, \frac{dy}{dx} = 3x^2 - 10x + 9$

(vi) Differentiate with respect to x, $y = \frac{3}{x^2} + \frac{4}{x}$

$= 3x^{-2} + 4x^{-1}$

These two functions can be differentiated by using the derivative of x^n

$\frac{dy}{dx} = 3 \text{ x } (-2x^{-3}) + 4 \text{ x } (-1x^{-2})$

$\frac{dy}{dx} = -\frac{6}{x^3} - \frac{4}{x^2}$

(vii) Differentiate with respect to $x, y = (2x - 5)(3x + 2)$

We cannot as yet differentiate a product so we multiply the factors.

Thus, $y = 6x^2 - 11x - 10$

$\frac{dy}{dx} = 12x - 11$

(viii) Differentiate with respect to x, $y = \frac{4x^3 + 5x}{x^2}$

We cannot as yet differentiate a quotient so we divide each term in the numerator by x^2

Thus, $y = 4x + \frac{5}{x}$

$= 4x + 5x^{-1}$

$\frac{dy}{dx} = 4 - 5x^{-2}$

$= 4 - \frac{5}{x^2}$

Unit: Differentiation	Topic: General Rules of Differentiation
Benchmark	**12.3.3.8** Calculate the derivatives of functions, including polynomial, rational, trigonometric, logarithmic, inverse, composite and exponential functions.

Learning Objective: By the end of the topic, students will be able to;

- recognise and apply the rules for differentiation that are classified as general rules,
- apply the general rules for differentiation to evaluate derivatives,
- apply the product rule and quotient rule to find derivative, and
- use the chain rule to find derivatives of composite functions.



Essential questions:

- What are the general rules for differentiation?
- When is the product rule applied?
- When is the quotient rule applied?
- Why do you use the chain rule to find derivatives?



Key Concepts(ASK-MT)	
Attitudes/Values	Confidently explain the general rules for differentiation.
Skills	Apply general rules to differentiate.
Knowledge	General rules for differentiation, product rule, quotient rule and the chain rule.
Mathematical Thinking	Think about how to evaluate the derivatives using the chain, product and quotient rules.

Content Background

General Rules of Differentiation

1. The Derivative of the Product of Two Functions

If $y = u \times v$ and u and v are functions of x,
Then $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ this can be written as $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$.

In words: The derivative of a product equals the second factor times the derivative of the first factor plus the first factor times the derivative of the second factor

Example 1: Find $\frac{dy}{dx}$ if $y = (4x+3)(2x^2-5)$

Solution: $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$= (4)(2x^2-5) + (4x+3)(4x)$

$= 24x^2 + 12x - 20$

Example 2: if $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$.
find $f'(2)$

Solution:

$f'(x) = v\frac{du}{dx} + u\frac{dv}{dx}$

$= (2)(3x^2-5x+1) + (2x-3)(6x-5)$

$= 6x^2 - 10x + 2 + 12x^2 - 28x + 15$

$= 18x^2 - 38x + 17$

Thus, $f'(2) = 72 - 76 + 17$

$= 13$

2. The Derivative of the Quotient of Two Functions

If $y = \frac{u}{v}$ and u and v are functions of x,

Then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$. This can also be written as $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

In words:

$$\text{The derivative of the quotient} = \frac{(\textit{denominator x derivative of numerator} - \textit{numerator x derivative of denominator})}{\textit{denominator squared}}$$

Example: Find the derivative of $\frac{2x-5}{x^2-3}$.

Solution: Let $y = \frac{2x-5}{x^2-3}$. Then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x^2-3)-(2x-5)(2x)}{(x^2-3)^2}$

$$= \frac{-2x^2+10x-6}{(x^2-3)^2}$$

$$= \frac{-2(x^2-5x+3)}{(x^2-3)^2}.$$

3. The Derivation of Composite Functions

Suppose we wish to differentiate $y = (x^2+5)^6$.
If we put $u = x^2+5$ then we have $y = u^6$ where $u = x^2+5$.

Now u is a function of x and y is a function of u.

Such an expression is referred to as function of a function or sometimes as a composite function. We can find the derivative of a composite function by using the following rule.

If $y = F(u)$ where $u = f(x)$, Then $\frac{dy}{dx} = \frac{dy}{du}\text{X}\frac{du}{dx}$

Example 1: Find $\frac{dy}{dx}$ if $y = (x^2+5)^6$

Solution: Let $u = x^2+5$ then $y = u^6$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 6u^5.2x$$

$$= 6(x^2+5)^5.2x$$

$$= 12x(x^2+5)^5$$

Example 2: Differentiate $y = \sqrt{2x+1}$

Solution: Let $u = 2x+1$ then $y = u^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}}.2$$

$$= u^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2x+1}}$$

Note: With practice the composite functions rule can be applied easily without writing in the substitution.

For instance, if $y = (x^2+5)^6$

$$\frac{dy}{dx} = 6(x^2+5)^5 \text{x } 2x = 12x(x^2+5)^5$$

4. The Derivative of Rational Functions

We will now use the result $\frac{d}{dx}(x^n) = nx^{n-1}$ for n rational. That is, if n is a rational number, and $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$

Examples

(1) If $y = x^{\frac{4}{3}}$
$\frac{dy}{dx} = \frac{4}{3}x^{\frac{1}{3}}$

(2) If $y = x^{-\frac{1}{4}}$
$\frac{dy}{dx} = -\frac{1}{4}x^{-\frac{5}{4}}$

(3) If $y = \sqrt{x^5}$
$Then\ y = x^{\frac{5}{2}}$
$\frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}}$
$= \frac{5\sqrt{x^3}}{2}$

(4) If $y = x\sqrt{x+x}$
Then $y = x^{\frac{3}{2}} + x$
$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + 1 = \frac{3\sqrt{x}}{2} + 1$

5. Product Rule

$y = f(x)g(x)$
$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$
$y = 2x(3x^{12} + 1)$

Can expand and apply the power rule or we can use the product rule.
Let $f(x) = 2x,\ g(x) = 3x^{12} + 1;\ f'(x) = 2,\ g'(x) = 36x^{11}$

$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x) = 2(3x^{12} + 1) + 2x(36x^{11})$
$\frac{dy}{dx} = 6x^{12} + 2 + 72x^{12}$
$\frac{dy}{dx} = 78x^{12} + 2$

6. Chain Rule for Composite Functions

$y = f(u), u = g(x).\quad \frac{dy}{dx} = g'(x)f'(u)$
$y = (2x+1)^5,\quad u = 2x+1,\ y = u^5, g'(x) = u' = 2, f'(u) = 5u^4$
$\frac{dy}{dx} = g'(x)f'(u) = 2(5u^4) = 10u^4 = 10(2x+1)^4$
$y = (2x+1)^5$: $Let\ u = 2x+1, then\ y = u^5$
$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$: $\quad \frac{dy}{du} = 5u^4;\quad \frac{du}{dx} = 2$: $\quad \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 5u^4.2 = 10u^4 = 10(2x+1)^4$

Unit: Differentiation	Topic: Derivative of Exponential and Logarithm Functions

Benchmark	**12.3.3.8** Calculate the derivatives of functions, including polynomial, rational, trigonometric, logarithmic, inverse, composite and exponential functions.

Learning Objective: By the end of the topic, students will be able to;

- find the derivative of exponential functions of the form $y = e^{ax}$ where *a* is any real constant and of the type $y = 2^x$, etc.
- find derivatives of logarithmic functions.



Essential questions:

- What is the general expression of an exponential function?
- How is the derivative to an exponential function evaluated?
- How can a logarithmic function be evaluated?



Key Concepts(ASK-MT)	
Attitudes/Values	Recognise the general form for a logarithmic function as well as an exponential function and provide reason how the derivatives to these functions are evaluated.
Skills	Evaluate the derivatives of exponential and logarithmic functions.
Knowledge	Derivatives to exponential and logarithmic functions.
Mathematical Thinking	Think about how to evaluate the derivatives of exponential and logarithmic functions.

Content Background

1. Differentiation of the Exponential and Logarithmic Functions

Derivative of $y = \ln x = \log_e x$

The natural logarithmic function is differentiable on (0, 1) and its derivative is the reciprocal function, that is,

$$\frac{d}{dx}\ln x = \frac{1}{x}, \quad x > 0$$

Example: Find each of the following y, find $\frac{dy}{dx}$.

(a) $y = 2 + 3\ln x$
(b) $y = \sin x.\ln x$
(c) $y = \sin x.\ln x$

Solution

(a) $\frac{dy}{dx} = \frac{d}{dx}(2 + 3\ln x)$

$= \frac{d}{dx}(2) + \frac{d}{dx}(3\ln x)$.....Term by term differntiation

$= 0 + 3.\frac{d}{dx}\ln x$.....Derivative of Constant & Constant Multiple rule

$= \frac{3}{x}$.....Derivative of $\ln x$.

(b) $\frac{dy}{dx} = \frac{d}{dx}(\sin x.\ln x)$

$= \sin x.\frac{d}{dx}\ln x + \ln x.\frac{d}{dx}\sin x$.....Product Rule

$= \frac{\sin x}{x} + \ln x.\cos x$.....*D*erivate of ln & sin rule

(c) $\frac{dy}{dx} = \frac{d}{dx}(x\ln 2x)$

$= x.\frac{d}{dx}(\ln 2x) + \ln 2x.\frac{d}{dx}x$.....*Product Rule*

$= x.\frac{d}{dx}(\ln 2 + \ln x) + \ln 2x$.....*Log property & power rule*

$= x.\left(0 + \frac{1}{x}\right) + \ln 2x$.....*Derivative of cons*tan*t & In*

$= 1 + \ln 2x$

2. Differentiation of Logarithm functions with other bases

To differentiate other logarithmic functions with different functions $\log_b x$ where $b \neq c$. we can use change of base formula:

$$\log_b m = \frac{\ln m}{\ln b}$$

Example: Find the derivative of $y = \log_2(5x^3)$.

Solution:

$\frac{dy}{dx} = \frac{d}{dx}\log_2(5x^3)$

$= \frac{d}{dx}\left(\frac{\ln 5x^3}{\ln 2}\right)$.....Change of base

$= \frac{1}{\ln 2}.\frac{d}{dx}(\ln 5 + 3\ln x)$.....Constant Multiple Rule and log Properties

$= \frac{1}{\ln 2}\left(\frac{d}{dx}\ln 5 + \frac{d}{dx}(3\ln x)\right)$.....Term by term differntiation

$= \frac{1}{\ln 2}\left(0 + 3.\frac{d}{dx}\ln x\right)$.....Derivative of the constant & Constant Multiple Rule

$= \frac{1}{\ln 2}.3.\frac{1}{x}$.....Derivative of the constant

$= \frac{3}{x\ln 2}$

To find the formula for the derivative of the exponential function exp, we use the fact that the functions ln and *exp* are inverses of each other and we need the following:

3. Differentiation of exponential functions

The exponential function is differentiable on $\mathbb{R}$ and its derivative is the function itself, that is,

$\frac{d}{dx}e^x = e^x, -\infty < x < \infty.$

Example: For each of the following y, find $\frac{dy}{dx}$.

(a). $y = 2e^x - \frac{3}{x}$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left(2e^x - \frac{3}{x}\right)$$
$$= \frac{d}{dx}(2e^x) - \frac{d}{dx}(3x^{-1})$$
$$= 2.\frac{d}{dx}e^x - 3.\frac{d}{dx}x^{-1}$$
$$= 2.e^x - 3.(-1).x^{-2}$$
$$= 2e^x + \frac{3}{x^2}$$

(b). $y = x^5 e^x$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}(x^5 e^5)$$
$$= e^x.\frac{d}{dx}x^5 + x^5.\frac{d}{dx}e^x$$
$$= e^x.5x^4 + x^5 e^x$$
$$= (x^5 + 5x^4)e^x$$
$$= x^4 e^x(x+5)$$

(c) $y = \frac{e^x}{\sin x}$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x}{\sin x}\right)$$
$$= \frac{\sin x.\frac{d}{dx}e^x - e^x.\frac{d}{dx}\sin x}{\sin^2 x}$$
$$= \frac{\sin x.e^x - e^x.\cos x}{\sin^2 x}$$
$$= \frac{e^x(\sin x - \cos x)}{\sin^2 x}$$

Unit: Differentiation	Topic: Differentiation of Trigonometric Functions
Benchmark	**12.3.3.8** Calculate the derivatives of functions, including polynomial, rational, trigonometric, logarithmic, inverse, composite and exponential functions.

Learning Objective: By the end of the topic, students will be able to;



- solve the derivatives of the three trigonometric functions,
- determine the derivative of $\tan(x)$ by using the quotient rule, and
- find the derivatives of the reciprocal trigonometric functions using the quotient rule.

Essential questions:



- How can the derivative of $y = \sin(x)$ be evaluated using the first principle of differentiation?
- What is the basis of finding the derivative of trigonometric functions and their reciprocals?

Key Concepts(ASK-MT)	
Attitudes/Values	Enjoy in evaluating trigonometric functions and the reciprocal trigonometric functions.
Skills	Evaluate the derivatives to trigonometric functions and the reciprocal functions.
Knowledge	Derivatives of the trigonometric functions.
Mathematical Thinking	Think about how to evaluate the derivatives of the trigonometric functions and the reciprocal trigonometric functions.

Content Background

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

1. Derivative of $y = \sin x$

The derivative of the above function is found from First Principle.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cosh+\sinh\cos x-\sin x}{h}$$
$$= \lim_{h\to 0}\frac{\sin x\cosh-\sin x+\sinh\cos x}{h} = \lim_{h\to 0}\frac{\sin x(\cosh-1)+\sinh\cos x}{h}$$
$$= (\sin x)\lim_{h\to 0}\frac{(\cosh-1)}{h}+(\cos x)\lim_{h\to 0}\frac{\sinh}{h} = \sin x(0)+\cos x(1)$$
$$= 0+\cos x$$
$$= \cos x$$

Remember: $\lim_{x\to 0}\frac{\sin x}{x} = 1$. So the derivative of $y = \sin x$ is $\cos x$.

Note: Using this result of the derivative of $\sin x$, all the other Trigonometric functions and their reciprocals can be found.

2. Derivative of $y = \cos x$.

We find the derivative using properties of trigonometry and derivative of $\sin(x)$.

$$y = \cos x = \sin\left(\frac{\pi}{2}-x\right)$$
$$\frac{dy}{dx} = \cos\left(\frac{\pi}{2}-x\right)(-1)$$
$$= -\cos\frac{\pi}{2}-x$$
$$= -\sin x.$$

3. Derivative of $y = \tan x$.

$$y = \tan x = \frac{\sin x}{\cos x} = \frac{u}{v}. \qquad v' = -\sin x$$

$$\frac{dy}{dx} = \frac{\cos x(\cos x) - (-\sin x)(\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x.$$

4. Derivative of Reciprocal Trigonometric Functions

Derivatives of the reciprocal trigonometric functions can be found by using the QUOTIENT Rule of differentiation.

$$y = \cot(x) = \frac{1}{\tan(x)} = \frac{1}{\frac{\sin x}{\cos x}} = \frac{\cos x}{\sin s}$$

$$y = \text{cosec}\, ant\,(x) = \csc(x) = \frac{1}{\sin x}$$

Example 1. Derivative of $y = \sin 3x$. This is a composite function. We will use the chain rule here.

$$y = \sin 3x. \text{ Let } u = 3x \text{ then } y = \sin u$$

$$\frac{du}{dx} = 3, \quad \frac{dy}{du} = \cos u$$

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{du}{dx} = \cos u.3 = 3\cos 3x.$$

Unit: Differentiation	Topic: Implicit Differentiation
Benchmark	**12.3.3.8** Calculate the derivatives of functions, including polynomial, rational, trigonometric, logarithmic, inverse, composite and exponential functions.

Learning Objective: By the end of the topic, students will be able to;



- use implicit differentiation to find the derivative of exponential functions of the forms $y = e^{ax}$ and $y = ax$, and
- use implicit differentiation to find the derivatives of logarithmic functions.

Essential questions:



- How can you use implicit differentiation to find derivatives to logarithmic functions?
- Can the method of implicit differentiation be applied to any other functions?

Key Concepts(ASK-MT)	
Attitudes/Values	Display confidence to explain the method of implicit differentiation and evaluate the derivatives to logarithmic functions.
Skills	Evaluate derivatives to logarithmic functions using implicit differentiation.
Knowledge	Method of Implicit Differentiation.
Mathematical Thinking	Think about how to evaluate derivatives using the method of implicit differentiation.

Content Background

1. Implicit Differentiation

Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation with respect to x.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx.

Example: (a) Find dy/dx given that $y^3 + y^2 - 5y - x^2 = -4$

Solution: (1) Differentiate both sides of the equation with respect to x.

$$y^3 + y^2 - 5y - x^2 = -4 \Rightarrow 3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

$$\Rightarrow \frac{dy}{dx}[\,3y^2 + 2y - 5\,] = 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}.$$

Example (b): Find the derivative of the equation of the circle with center at the origin and radius of 2 units by implicit differentiation.

$$x^2 + y^2 = 4$$

$$2x + 2y\frac{dy}{dx} = 0 \Leftrightarrow 2y\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

Apply the normal rules of differentiation on all terms and wherever there is y write dy/dx besides and eventually solve for dy/dx.

2. Finding the derivative of $y = e^{2x}$ by implicit differentiation.

$y = e^{2x}$: take ln both sides;

Recall: Derivative of $y = \ln x$ is $\frac{1}{x}$.

(a) Find the derivative of $y = 2^x$.. First take $\log_e = \text{In}$ both sides. We obtain:

$$\text{lny} = \ln 2^x \Leftrightarrow \text{lny} = \text{xln2} \Leftrightarrow \frac{1}{y}\frac{dy}{dx} = \ln 2 \Leftrightarrow \frac{dy}{dx} = \text{yln2} = 2^x \ln 2$$

(b) Find the derivative of $y = 3^{2x-1}$. Same as above.

$$\ln y = \ln 3^{(2x-1)} \Leftrightarrow \ln y = (2x-1)\ln 3 \Leftrightarrow \ln y = 2x\ln 3 - \ln 3 \Leftrightarrow \frac{1}{y}\frac{dy}{dx} = 2\ln 3 \Leftrightarrow \frac{dy}{dx} = (2\ln 3)y \Leftrightarrow (2\ln 3)(3^{(2x-1)})$$

(c) Find the derivative of $y = \log_5(2x)$. We will apply two strategies here to find the derivative.

Strategy 1: $y = \log_5(2x) = \frac{\ln(2x)}{\ln 5} = \frac{1}{\ln 5}.\ln(2x) : \frac{dy}{dx} = \frac{1}{\ln 5}.\frac{2}{2x} = \frac{1}{x\ln 5}$

Change of base formula.

Strategy 2:

$$y = \log_5(2x) \Leftrightarrow 5^y = 2x \Leftrightarrow \ln 5^y = \ln(2x) \Leftrightarrow y\ln 5 = \ln(2x) : \ln 5\frac{dy}{dx} = \frac{2}{2x} \Leftrightarrow \ln 5\frac{dy}{dx} = \frac{1}{x} \Leftrightarrow \frac{dy}{dx} = \frac{1}{x\ln 5}$$

Express in exponential form and take *ln* both sides and proceed as above examples.

Unit: Differentiation	Topic: Tangents, Normal Maxima, Minima, and Stationary points
Benchmark	**12.3.3.9** Calculate tangents, normal and stationary points on curves.

Learning Objective: By the end of the topic, students will be able to;

- define a tangent and a normal line,
- find the equations of tangents and normal to graphs of a function at any given point,
- find the coordinates of the point on the graph if the gradient of the tangent or normal is given,
- understand the stationary point(s) in terms of the graph and tangents (gradients),
- calculate stationary point(s) of a given function, and
- determine from the sign of the derivative (gradient) which is maximum and minimum.



Essential questions:

- Is there a difference between a normal and a tangent?
- How are the equations to a normal and a tangent evaluated?
- Describe maxima, minima and the stationary points?

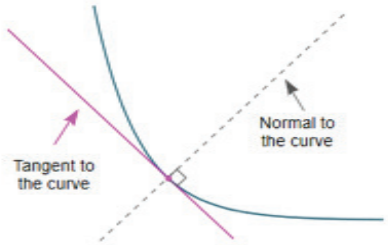


Key Concepts(ASK-MT)	
Attitudes/Values	Confidently describe and evaluate the stationary points and other major points on a curve through observing the behavior of the curve.
Skills	Evaluating stationary points, maxima, minima, tangents and normals.
Knowledge	Stationary points, maxima, minima, tangent and normal.
Mathematical Thinking	Think about how to evaluate the equations to a tangent and a normal.

Content Background

1. Tangent and Normal to Curve

The graph below illustrates the tangent and the normal to a curve at a point.



The gradient of the tangent is equal to the gradient of the curve at the point of tangency. The normal is perpendicular to the tangent at the same point of tangency.

Example: A function is given as $y = 2x^2 - 3x + 2$. Find the equation of the tangent and normal at $x = 1$.

Solution: $\frac{dy}{dx} = 4x - 3$: gradient of tangent at $x = 1$ is $4(1) - 3 = 4 - 3 = 1$. .

Equation of tangent: $y = mx + c : y = (1)x + c : At\ x = 1, y = 2(1)^2 - 3(1) + 2 = 1.$

$(1,1) : y = mx + c :\ y = x + c : 1 = 1 + c : c = 0.\ y = x$

Equation of normal: gradient =-1: $y = -x + c: 1 = -1 + c: \quad c = 2. \quad y = -x + 2$.

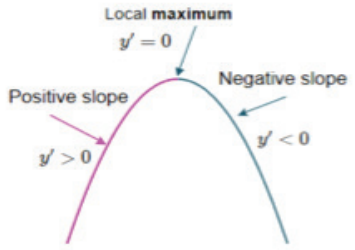
So the equation of the tangent is $y = x$ and the normal is $y = -x + 2$.

Example: The gradient of the tangent to the curve $y = x^2 - 3x$ is 3. Find the coordinates at which the tangent meets the curve.

Solution: $\frac{dy}{dx} = 2x - 3. \qquad 2x - 3 = 3 \Leftrightarrow 2x = 6 \Leftrightarrow x = 3: \; y = 3^2 - 3(3) = 0 : point(3,0).$

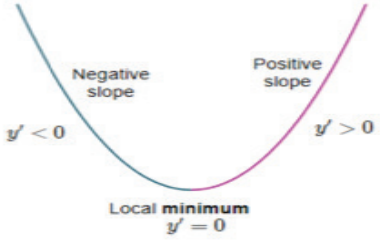
2. Finding Maxima and Minima

Local Maximum – A local maximum occurs when $y' = 0$ and y' changes sign from positive to negative (as we go left to right).



Curve showing portion with positive slope in pink, and negative slope in green.

Local Minima – A local maximum occurs when $y' = 0$ and y' changes sign from negative to positive



Curve showing positive slope in pink, and negative slope in green.

Example: Find the stationary point(s) of this function $y = 2x^2 - 3x + 2.$.

Solution: $\frac{dy}{dx} = 6x^2 - 12x: \; 6x^2 - 12x = 0 \Leftrightarrow x^2 - 2x = 0 \Leftrightarrow x(x-2) = 0 \Leftrightarrow x = 0 \, or \, 2.$

$$x = 0 : y = 2(0)^3 - 6(0)^2 + 2 = 2 : \; x = 2 : y = 2(2)^3 - 6(2)^2 + 2 = -6.$$

Two stationary points are (0,2) and (2,-6). Now we will determine which is maximum and which is minimum turning point.

Point (0,2) is a maximum turning point as indicated by the sign of the derivative (+++ 0 - - -). Point (2,-6) is minimum turning point as indicated by the sign of the derivative (- - - 0 + + +).

Refer to the tables given below.

x left	dy/dx	dy/dx at x=0	x right	dy/dx
-0.1	1.26	0	0.1	-1.14
-0.01	0.1206	0	0.01	-0.1194
-0.001	0.012006	0	0.001	-0.01199
-0.0001	0.0012	0	0.0001	-0.0012
	+	0		-

x left	dy/dx	dy/dx at x=0	x right	dy/dx
1.8	-2.16	0	2.1	1.26
1.9	-1.14	0	2.01	0.1206
1.99	-0.1194	0	2.001	0.012006
1.999	-0.01199	0	2.0001	0.0012
	-	0		+

Unit: Differentiation	Topic: The Second Derivative

Benchmark	**12.3.3.10** Solve problems on maxima and minima, and practical problems on rate of change.

Learning Objective: By the end of the topic, students will be able to;



- find second derivative of functions,
- use the second derivative test to determine which stationary point is maximum or minimum,
- use the second derivative to determine the point of inflection/inflexion , (point where the curvature of the graph changes). *i.e.* when $f''(x) = 0$,
- State the domain in which the function is increasing and decreasing, and
- sketch the graph of a function by using the following information; turning points,intercepts, point of inflection, a few points and by studying the equation of the given function.

Essential questions:



- Why is there a second derivative?
- What happens when asked to evaluate the second derivative to a function?
- What is the meaning of the second derivative?

Key Concepts(ASK-MT)	
Attitudes/Values	Reasonably explain the significance of taking the second derivative to a function.
Skills	Evaluate the second derivative to functions.
Knowledge	The second derivative.
Mathematical Thinking	Think about how to evaluate the second derivative to a function.

Content Background

1. The second derivative

At a critical point of a differentiable function, the First Derivative Test tells us whether there is a local maximum or a local minimum, or whether the graph just continues to rise or fall there. In this topic we see how the second derivative gives information about the way the graph of a differentiable function bends or turns. This additional information enables us to capture key aspects of the behavior of a function and its graph, and then present these features in a sketch of the graph. Using the functional notation, the second derivative of the function f is written as f''. Using the Leibniz notation, the second derivative is written as $\frac{d^2y}{dx^2}$, where y is the function of x.

2. Concave Up and Concave down

If $\frac{d^2y}{dx^2} > 0$, the curve will have a minimum-type shape (called concave up)

$$y'' = \frac{d^2y}{dx^2} > 0$$

Concave up curve, where $y'' > 0$ for all x

If $\frac{d^2y}{dx^2} < 0$, the curve will have a maximum-type shape (called concave down)

$$y'' = \frac{d^2y}{dx^2} < 0$$

Example 1

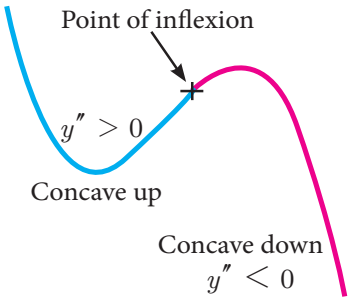
The curve $y = x^2 + 3x - 2$ has $\frac{dy}{dx} = 2x + 3$. Now $\frac{d^2y}{dx^2} > 2$ and of course, this is > 0 for all values of x. So it has a concave up shape for all x.

Example 2

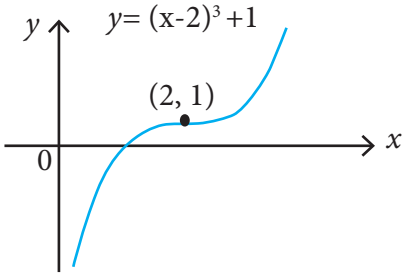
The curve $y = x^3 - 2x + 5$ has $\frac{dy}{dx} = 3x^2 - 2$. . The second derivative is $\frac{d^2y}{dx^2} > 6x$ and this is < 0 for all values of $x < 0$. So the curve has a concave down shape for all $x < 0$ (and it is concave if $x > 0$).

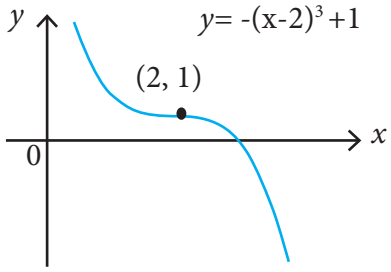
3. Points of Inflection

A point of inflection will occur when $\frac{d^2y}{dx^2} = 0$ and when there is a change in sign (from plus → minus or minus → plus) or from concave up to concave down verse versa of $\frac{d^2y}{dx^2}$.



The graph of $y = (x-2)^3 + 1$ and $y = -(x-2)^3 + 1$ are shown below. The point (2, 1) is a point of inflexion for each of these graphs. In fact, the point (2, 1) is a stationary point of inflexion for each of these graphs.





4. Curve Sketching

Locate and describe the stationary points of $f(x) = 4^4 - 8x^2$ and sketch its graph

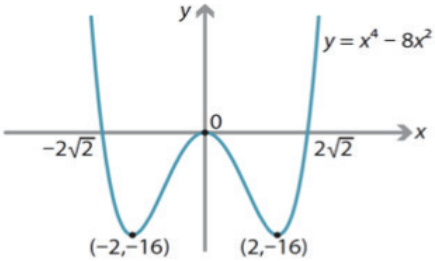
Solution: The first derivative is $f'(x)$ $^{-}$ $4x^3 - 16x$ and the second derivative is $f''(x) = 12x^2 - 16$. We find the stationary points by solving $f'(x) = 0$:

$4x^3 - 16x = 0 \Leftrightarrow 4x(x^2 - 4) = 0$

We can now check the local minimum and maximum points using the second derivative test, by considering the values of the second derivative:

$f''(-2) = 32 > 0$,so there is a local minimum at $x = -2$.
$f''(0) = -16 < 0$, so there is a local maximum at $x = 0$
$f''(2) = 32 > 0$,so there is a local minimum at $x = 2$.

Now we have enough information to complete a good sketch if we also observe that the x-intercepts are $0, 2\sqrt{2}$ and $-2\sqrt{2}$.



Unit: Differentiation	Topic : Applied Optimization Problems
Benchmark	**12.3.3.10** Solve problems on maxima and minima, and practical problems on rate of change.

Learning Objective: By the end of the topic, students will be able to;

- solve application problems in terms of optimization.

Essential questions:

- What are some real life problems that can be solved using calculus?

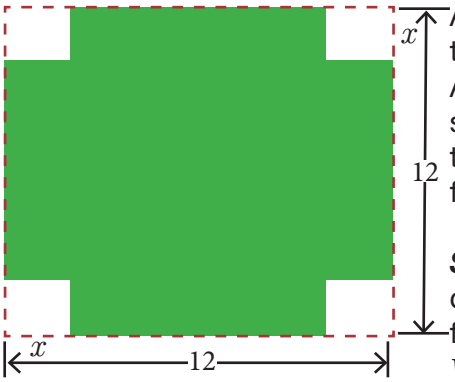
Key Concepts(ASK-MT)	
Attitudes/Values	Appreciate the application of differential calculus in solving real problems.
Skills	Solve real life problems using differential calculus.
Knowledge	Optimization – application of differential calculus.
Mathematical Thinking	Think about how to solve real life problems involving differential calculus.

Content Background

Applied Optimization Problems

To optimize something means to maximize or minimize some aspect of it. What are the dimensions of a rectangle with fixed perimeter having maximum area? What is the least expensive shape for a cylindrical can? What is the size of the most profitable production run? Differential calculus is a powerful tool for solving problems that call for maximizing or minimizing a function. In this topic we solve a variety of optimization problems from business, mathematics, physics, and economics.

1. Examples from Business and Industry



An open box made by cutting the corners from a square sheet of tin. What size corners maximize the box's volume?
An open-top box is to be made by cutting small congruent squares from the corners of a 12 inches by 12 inches. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution: We start with a picture (Figure-(a)). In the figure, the corner squares are x inch on a side. The volume of the box is a function of this variable:

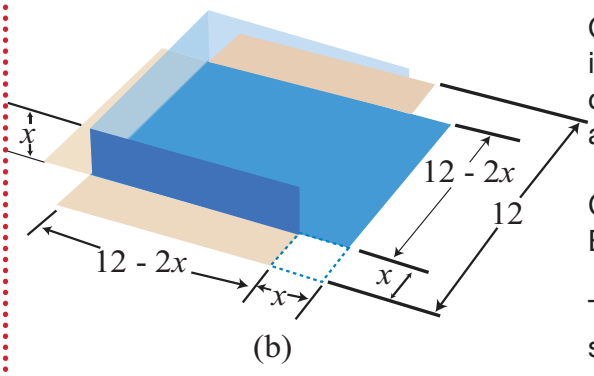
$V(x) = x(12-2x)^2 = 144x - 48x^2 + 4x^3$

Since the sides of the sheet of tin are only 12 in. $x \le 6$ long, and the domain of V is the interval $0 \le x \le 6$.

A graph of V *(Figure - (b))* suggests a minimum value of 0 at $x = 0$ and $x = 6$ and a maximum near $x = 6$.

To learn more, we examine the first derivative of V with respect to x:

$\frac{dV}{dx} = 144 - 96x + 12x^2 = 12(12 - 8x - x^2) = 12(2-x)(6-x)$

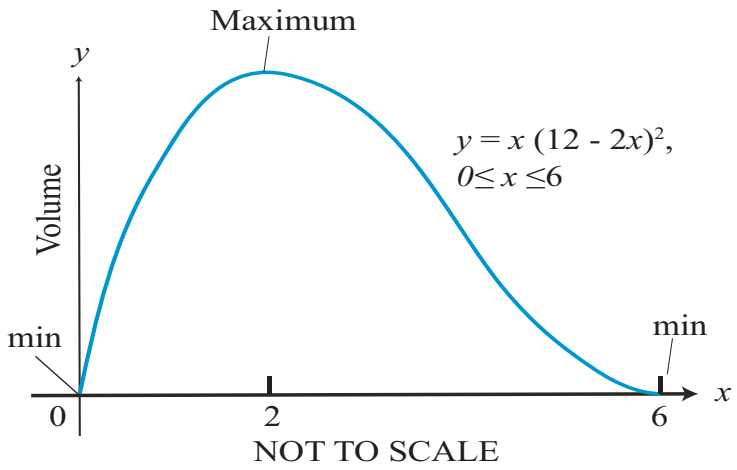


(b)

Of the two zeros, $x = 2$ and $x = 6$, only $x = 2$ lies in the interior of the function's domain and makes the critical-point list. The values of V at this one critical point and two endpoints are

Critical-point value: $V(2) = 128$
Endpoint values: $V(2) = 0, \quad V(6) = 0$

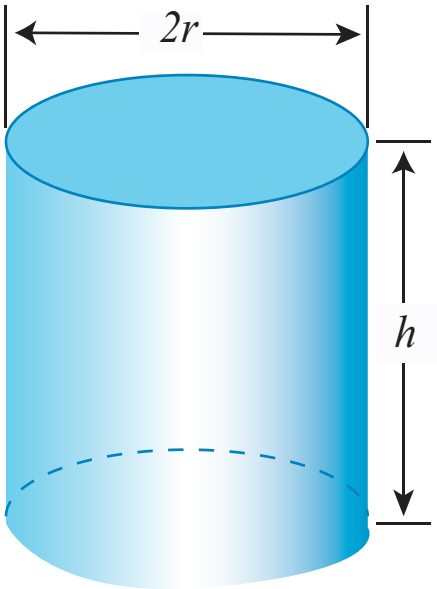
The maximum volume is 128 inches ^{3}. The cutout squares should be 2 in. on a side.



The volume of the box in Figure (b) graphed as a function of x.

You have been asked to design a 1-liter can shaped like a right circular cylinder *(Figure 4.34)*. What dimensions will use the least material?

Solution: Volume of can: If r and h are measured in centimeters, then the volume of the can in cubic centimeters is: $1Litter = 100cm^3$ Note: $1Litter = 100cm^3$



Surface area of can: $A = 2\pi r^2 + 2\pi rh$

How can we interpret the phrase "least material"? First, it is customary to ignore the thickness of the material and the waste in manufacturing. Then we ask for dimensions r and h that make the total surface area as small as possible while satisfying the constraint $\pi r^2 h = 1000$.

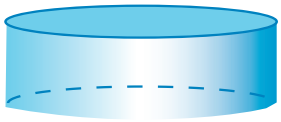
To express the surface area as a function of one variable, we solve for one of the variables in $\pi r^2 h = 1000$ and substitute that expression into the surface area formula. Solving for h is easier:

$$h = \frac{1000}{\pi r^2},$$

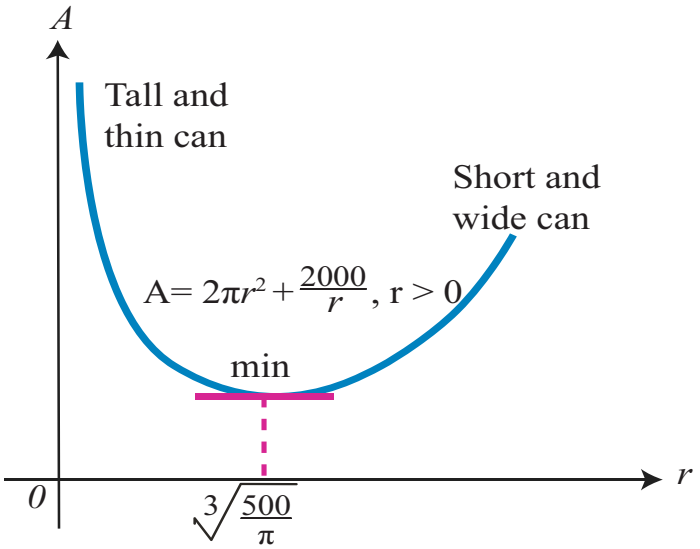
Thus $A = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$
Our goal is to find a value $r > 0$ that minimizes the value of A. The figure suggests that such a value exists.



Tall and thin



Short and wide



The graph of $A = 2r^2 + \frac{2000}{r}$ is concave up.

Notice from the graph that for small r (a tall thin container, like a piece of pipe), the term $\frac{2000}{r}$ dominates and A is large. For large r (a short wide container, like a pizza pan), the term $2\pi r^2$ dominates and A again is large. Since A is differentiable on r>0, an interval with no endpoints, it can have a minimum value only where its first derivative is zero.

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

$$0 = 4\pi r - \frac{2000}{r^2}$$

$$4\pi r^3 = 2000$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$$

What happens at $= \sqrt[3]{\frac{500}{\pi}}$?

The second derivative $\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}$ is positive throughout the domain of A. The graph is therefore everywhere concave up and the value of A at $r = \sqrt[3]{\frac{500}{\pi}}$ an absolute minimum. The corresponding value of h (after a little algebra) is $h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$

The $1-L$ can that uses the least material has height equal to the diameter, here with $r \approx 5.42cm$ and $h \approx 10.84cm$.

Unit: Integration	Topic: Anti-Derivatives and Indefinite Integrals
Benchmark	**12.3.3.11** Define theorems and rules of integration.
	12.3.3.12 Calculate the anti-derivatives of functions, including polynomials, rational, trigonometric, inverse, composite and exponential functions

Learning Objective: By the end of the topic, students will be able to;



- define anti-derivatives and find anti-derivatives of simple functions,
- find indefinite integrals of a function.

Essential questions:



- What is the relationship between anti-derivative and integral?
- What is the core objective of anti-differentiation or Integration?

Key Concepts(ASK-MT)	
Attitudes/Values	Clearly understanding and reasoning out the goal of integration so that it makes more sense when integrating functions.
Skills	Provide logical reasons to explain integral calculus
Knowledge	Anti-Derivatives/Integrals
Mathematical Thinking	Explanation and evaluation of integrals using the basic rules

Content Background

1. Anti-Derivatives

Previously $f(x)$ was given and we found $f'(x)$. Now we are given $f(x)$ and asked to find $f'(x)$ such that $f'(x)=f(x)$.
Definition: A function F is called the antiderivative of f if $f'(x)=f(x)$..
Because $\frac{d}{dx}[x^2]=2x,$ therefore $F(x)=x^2$ is an antiderative of $f(x)=2x$. Similarly f(x)=x^3 is the antiderative of f(x)=$3x^2$. The antiderative of 2x can be any of these; x^2+1,x^2-1,x^2+5, etc.

Hence in general the antiderative can be F(x)+C where C is any constant, is the set of all anti-derivatives of f(x)=2x.

We may regard integration as the reverse of differentiation. So if we have a table of derivatives, we can read it backwards as a table of anti-derivatives. When we do this, we often need to deal with constants which arise in the process of differentiation.

Suppose that F(x) and G(x) are anti-derivatives of a function f (x). Then F(x) and G(x) differ only by a constant. That is, F(x) = G(x)+C for some constant C.

Note: To find all antiderivatives of a function, it is necessary only to find one antiderivative and then add a generic constant to it.

The Indefinite Integral

So for the function f (x) = 2x, since F(x) = x2 is an antiderivative then all antiderivatives of f (x) are of

the form $F(x) = x^2 + C$, where C is a generic constant. Thus, functions do not have just one antiderivative but a whole family of antiderivatives, all differing only by a constant. The following notation makes all this easier to express:

The indefinite integral of a function $f(x)$ is denoted by $\int f(x)dx$ (without upper and lower boundaries/limit) and represents the entire family of antiderivatives of $f(x)$.

Finding (or evaluating) the indefinite integral of a function is called integrating the function, and integration is antidifferentiation.

Important Results

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$
2. $\int kn\ dx = k\int x\ dx$
3. $\int dx = \int 1dx = x + C$
4. $\int \{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx$
5. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$

Examples

i. Find $\int (x^3 - 2x^2 + 1)dx$.

Solution

$$\int (x^3 - 2x^2 + 1)dx = \int x^3 dx - 2\int x^2 dx + \int dx$$
$$= \frac{x^4}{4} - \frac{2x^3}{3} + x + C$$

Note: Only one constant is needed as it represents the sum of the individual constants.

ii. Find the indefinite integral of $x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

Solution

$$\int f(x)dx(x^{\frac{1}{2}} + x^{-\frac{1}{2}})dx = \int x^{\frac{1}{2}}dx + \int x^{-\frac{1}{2}}dx$$
$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$
$$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C.$$

iii) Find the indefinite integral $\int \frac{x^3 - 2}{x^2}dx$

Solution

$$\int \frac{x^3 - 2}{x^2}dx = \int \left(x - \frac{2}{x^2}\right)dx$$
$$= \frac{x^2}{2} + \frac{2}{x} + C.$$

iv. Find $\int (2x+3)^4 dx$

Solution $\int (2x+3)^4 dx = \frac{(2x+3)^5}{x \text{ x } 5} + C$

$$= \frac{(2x+3)^5}{10} + C$$

Unit: Integration	Topic: Integrals of Trigonometric and Exponential Functions
Benchmark	**12.3.3.12** Calculate the anti-derivatives of functions, including polynomial, rational, trigonometric, inverse, composite and exponential functions.

Learning Objective: By the end of the topic, students will be able to;

- find the integrals of $\sin x$ and $\cos x$,
- find the integrals of differentiable trigonometric functions using differentiation techniques.



Essential questions:

- How are the integrals to trigonometric and exponential functions calculated?



Key Concepts(ASK-MT)	
Attitudes/Values	Appreciate the method of finding integrals of trigonometric and exponential functions.
Skills	Evaluate and explain integrals of trigonometry and exponential functions.
Knowledge	Integration of trigonometric and exponential functions
Mathematical Thinking	Think about how to evaluate the integrals to trigonometric and exponential functions.

Content Background

1. Integral of Trigonometric Functions

The following primitive or indefinite integrals are just re-statements of the corresponding derivative formulas for the six basic trigonometric functions.

$$\int \cos x\, dx = \sin x + c$$

$$\int \sin x\, dx = -\cos x + c$$

$$\int \sec^2 x dx = \tan x + c$$

$$\int \sec x \tan x\, dx = \sec x + c$$

$$\int \csc x \cot x\, dx = -\csc + c$$

$$\int \csc^2 x dx = -\cot x + c$$

More general results

$$\int \cos ax dx = \frac{1}{a}\sin ax + c$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax + c$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax + c$$

Examples

(a) Find the primitive function of $\cos 3x$.

Solution: $\int \cos 3x dx = \frac{1}{3}\sin 3x + c$

(b) Find $\int \sec^2(4x+3)dx = \frac{1}{4}\tan(4x+3)+C$

Solution: $\int \sec^2(4x+3)dx = \frac{1}{4}\tan(4x+3)+C$

2. Integral of Exponential Functions

Example (i): Show that if $y = e^{5x+1}$, then $\frac{dx}{dy} = 5e^{5x+1}$ .Hence find $\int e^{5x+1}dx$.

Solution: If $y = e^{5x+1}$, , then $\frac{dx}{dy} = 5e^{5x+1}$, hence $\int 5e^{5x+1}dx = \frac{1}{5}e^{5x+1}+C$

So $\int e^{5x+1}dx = \frac{1}{5}e^{5x+1}+C$

Example (ii): Find $\int_1^2 xe^{x^2}dx$

Solution: Note that $\frac{d}{dx}(e^{x^2}) = 2xe^{x^2}$

So $\int 2xe^{x^2}dx = e^{x^2}+C$

$$\int_1^2 xe^{x^2}dx = \frac{1}{2}\int_1^2 2xe^{x^2}dx$$

Thus $= \frac{1}{2}[e^{x^2}]_1^2$

$= \frac{1}{2}(e^4 - e).$

Note: It is important to remember derivative laws of exponential and or logarithm functions to find their derivative.

(i) $\frac{d}{dx}(a^x) = a^x \ln(a)$

(ii) $\frac{d}{dx}(\ln(x)) = \frac{1}{x}, x > 0$

(iii) $\frac{d}{dx}(\ln|x|) = \frac{1}{x}, x \neq 0$

(iv) $(\log_a(x)) = \frac{1}{x\ln a}, x > 0$

3. The Derivative of $y = \log_e x$ and Integral of $\frac{1}{x}$

Given $y = \log_e x$

Then this means $x = e^y$ from which $\frac{dy}{dx} = e^y$

$$\frac{dy}{dx} = \frac{1}{\frac{dy}{dx}}$$

But $= \frac{1}{e^y}$

$= \frac{1}{x}$

This is a very special result. If $y = \log_e x$ then $\frac{dy}{dx} = \frac{1}{x}$

Hence, also $\int \frac{1}{x}dx = \log_e x + c = \ln x + c$.

Unit: Integration	Topic : Definite Integrals
Benchmark	**12.3.3.12** Calculate the anti-derivatives of functions, including polynomial, rational, trigonometric, inverse, composite and exponential functions.

Learning Objective: By the end of the topic, students will be able to;

- define indefinite integral and evaluate definite integrals of functions.



Essential questions:

- What does it mean to evaluate definite integral?
- What is the fundamental theorem of calculus?



Key Concepts(ASK-MT)	
Attitudes/Values	Confidently integrate within the limits given and be able to explain the integrals.
Skills	Apply the rules of definite integrals.
Knowledge	Definite integrals.
Mathematical Thinking	Think about how to evaluate definite integral.

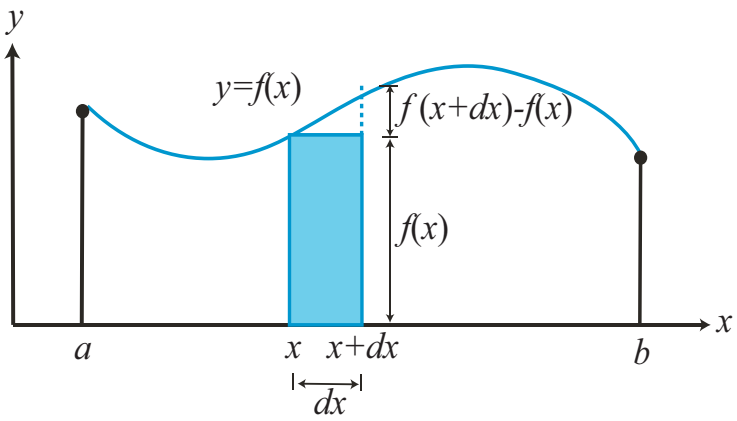
Content Background

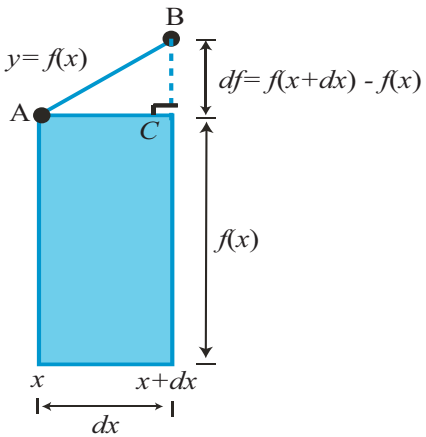
1. Definite Integrals

Recall from the last topic that the integral sign in the indefinite integral $\int f(x)dx$ represents a summation of the infinitesimals $f(x)dx = dF$ for an antiderivative $F(x)$ of $f(x)$. Why is the term "indefinite" used? Because the summation is indefinite: the x in $f(x)dx$ is defined generically, meaning "x in general," that is, not for x in a specific range of values. The same summation over a specific, definite range of values of x, say, over an interval $[a,b]$, is a different type of integral:

> The definite integral of the function $f(x)$ over an integral $[a,b]$ is denoted by
> $$\int_a^b f(x)dx$$
> and represents the sum of the infinitesimals $f(x)dx$ for all over x in $[a,b]$.

An indefinite integral yields a generic function, whereas a definite integral yields either a number or a specific function. There are many ways to calculate the specific summation in a definite integral, one of which is motivated by a geometric interpretation of the infinitesimal $f(x)dx$ as the area of a rectangle, as in the figure below:





The infinitesimal $f(x)dx$ as the area of a rectangle Area under the curve $y = f(x)$ over $[x, x + dx]$

The shaded rectangle in the above picture has height $f(x)$ and width dx, and so its area is $f(x)dx$. In fact, it appears that that area is just a little bit smaller than the area under the curve $y=f(x)$ and above the x-axis between x and $x+dx$; there is a small gap between the curve and the top of the rectangle, accounting for the difference in the area. However, the area of that gap turns out to be zero, as shown by the figure on the right.

Over the infinitesimal interval $[x,x+dx]$ the curve $y=f(x)$ shown in Figure on the left is, by the Micro straightness Property, a straight line, as shown in the figure on the right. Thus, the part of the area between the curve and the x-axis over the interval $[x,x+dx]$ consists of two parts: the area $f(x)dx$ of the shaded rectangle and the area of the right triangle ΔABC, both of which are shown in by the figure on the right. But the area of ΔABC is zero:

Area of $\Delta ABC=\frac{1}{2}\,Base\ X\ Height=\frac{1}{2}(dx)(df)(f'(x)dx)=\frac{1}{2}f'(x)(dx)^2=0$

The function f shown in Figure on the right is increasing at x, but a similar argument could be made if f were decreasing at x. Hence, the area between the curve $y=f(x)$ and the x-axis comes solely from the rectangles with area $f(x)dx$, as x varies from a to b. The sum of all those rectangular areas, however, is the definite integral of $f(x)$ over $[a,b]$, so that the definite integral can be interpreted as an area:

For a function $f(x)\geq 0$, the area under the curve $y=f(x)$ between $x=a$ and $x=b$, denoted by A, is given by

$$\boldsymbol{A=\int_a^b f(x)dx}$$

and represents the area of the region R bounded above by $y=f(x)$, bounded below by the x-axis, and bounded on the sides by $x=a$ and $x=b$ (with $a<b$).

The area A of the region R equals $\int_a^b f(x)dx$

Note: In the definite integral $\int_a^b f(x)dx$ the numbers a and b are called the **limits of integration**, with a being the **lower limit of integration** and b the **upper limit of integration**. The function $f(x)$ being integrated is called the **integrand,** in both a definite integral and indefinite integral.

4. Rules for Definite Integrals

If F is an antiderivative of f on $[a,b]$,i.e. $F'(x)=f(x)$ for all x in $[a,b]$, then

$$\int_a^b \boldsymbol{f(x)=F(b)-F(a)}$$

The following rules for definite integrals are a consequence of the corresponding rules for indefinite integrals:

Let f and g be continuous functions on $[a,b]$ and let k be a constant. Then:

1. $\int_a^b \boldsymbol{f(x)dx=k\int_a^b f(x)dx}$

2. $\int_a^b f(x)+g(x)dx=\int_a^b f(x)dx+\int_a^b g(x)dx$

3. $\int_a^b f(x)-g(x)dx=\int_a^b f(x)dx-\int_a^b g(x)dx$

Example

(a) Calculate $\int_1^2 x^2\,dx$

Solution: $\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$

(b) Evaluate the following definite integrals

(i) $\int_1^3 (3x^2 + 2x + 1)\,dx$ **(ii)** $\int_1^4 \left(x + \frac{1}{x^2}\right)dx$ **(iii)** $\int_{-1}^2 (x-1)^2\,dx$

Solutions:

(i) $\int_1^3 (3x^2 + 2x + 1)\,dx$

$= [x^3 + x^2 + x]_1^3$

$= (3^3 + 3^2 + 3) - (1^3 + 1^2 + 1)$

$= 36$

(ii) $\int_1^4 \left(x + \frac{1}{x^2}\right)dx$

$= \int_1^4 (x + x^{-2})\,dx$

$= \left[\frac{x^2}{2} - \frac{1}{x}\right]_1^4$

$= \left(8 - \frac{1}{4}\right) - \left(\frac{1}{2} - 1\right)$

$= 7\frac{3}{4} - \left(-\frac{1}{2}\right)$

$= 8\frac{1}{4}$

(iii) $\int_{-1}^2 (x-1)^2\,dx$

$= \int_{-1}^2 (x^2 - 2x + 1)\,dx$

$= \left[\frac{x^3}{3} - x^2 + x\right]_{-1}^2$

$= \left(\frac{8}{3} - 4 + 2\right) - \left(-\frac{1}{3} - 1 - 1\right)$

$= 3$

(c) Evaluate $\int_0^{\frac{\pi}{2}} (1 + \sin 2x)\,dx.$

Solution: $\int_0^{\frac{\pi}{2}} (1 + \sin 2x)\,dx = \left[x - \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$

$= \left[\frac{\pi}{2} - \frac{1}{2}(-1)\right] - \left[0 - \frac{1}{2}(1)\right]$

$= \frac{\pi}{2} + \frac{1}{2} + \frac{1}{2}$

$= \frac{\pi}{2} + 1$

$\int_0^3 (2x+3)\,dx = x^2 + 3x$ .at $x = 3$:we obtain 18.at $x = 0$,we obtain 0. The difference is $18 - 0 = 18.$

(e) $\int_{\frac{\pi}{2}}^{2\pi} \sin 2x\,dx = \frac{1}{2}\cos 2x.$ At $\frac{\pi}{2}$,we get $\frac{1}{2}\cos\left(2\frac{\pi}{2}\right) = -0.5\cos(\pi) = -0.5(-1) = 0.5 = \frac{1}{2}$

At 2π ,we get $-0.5\cos(2(2\pi)) = -0.5\cos(4\pi) = -0.5(-1) = 0.5 = \frac{1}{2}$

Upper value - lower value $= 0.5 - (0.5) = 0.5 - 0.5 = 0$.

(f) $\int_{-1}^2 (x^2 - 3x)\,dx = \frac{1}{3}x^3 - \frac{3}{2}x^2$. Upper bound: $\frac{1}{3}(2)^3 - 1.5(2)^2 = \frac{8}{3} - 6 = -\frac{10}{3}$.

Lower Bound: $\frac{1}{3}(-1)^3 - 1.5(-1)^2 = -\frac{11}{6}$

Difference $= UB - LB = -\frac{10}{3} - \left(-\frac{11}{6}\right) = -\frac{3}{2} = -1.5$

Unit: Integration	Topic: Area of Curves
Benchmark	**12.3.3.13** Solve problems on applications of integration.

Learning Objective: By the end of the topic, students will be able to;

- calculate area enclosed by a straight line (linear function) and the x-y coordinate system,
- calculate area enclosed by a curve and the x-axis,
- calculate area enclosed by a curve and a straight line,
- calculate area enclosed by two curves, and
- calculate areas enclosed/bounded by curves below the x-axis.



Essential questions:

- Does the position of the curve make any difference to the area?
- What is the significance of calculating area under curves?
- How is the area under the curve calculated?



Key Concepts(ASK-MT)	
Attitudes/Values	Confidently apply integration techniques to find the area under a curve.
Skills	Apply integration techniques to find the area under a curve.
Knowledge	Application to integration.
Mathematical Thinking	Think about how to calculate area under a curve by using integration techniques.

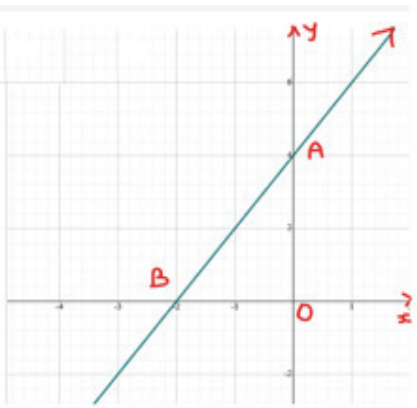
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Integration can be used to calculate areas. In simple cases, the area is given by a single definite integral. But sometimes the integral gives a negative answer which is minus the area, and in more complicated cased the correct answer can be obtained only by splitting the area into several parts and adding or subtracting the appropriate integrals. You should be able to;

- Find area between a curve, the x-xis and two given ordinates;
- Find area between a curve and, the x-axis, where the ordinates are given by the points where the curve crosses the axis;
- Find the area between two curves.

1. Area bounded by a line and the axes

The graph of $y = 2x + 4$ is given on the right. The line cuts off -2 and 4 on the x and y axes respectively. A perfect right triangle OAB is formed. The area of this triangle can be calculated using the formula $A = \frac{bh}{2} = \frac{2 \times 4}{2} = 4$ square units. We can also use calculus (integration) to find the area of the triangle.



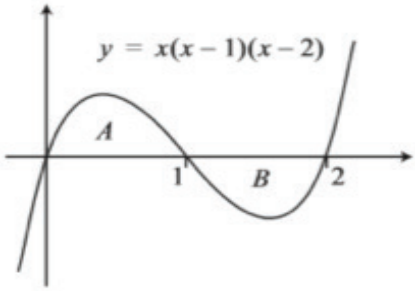
$$\int_{-2}^{0} (2x + 4)dx = x^2 + 4x.$$

Upper: $(0)^2 + 4(0) = 0$
Lower: $(-2)^2 + 4(-2) = -4$
Area = $U - L = 0 - (-4) = 4$ square units.

We can obtain the area between a curve, the x-axis, and specific ordinates (that is, values of x), by using integration.

2. The area between a curve and the x-axis

Let us begin by exploring the following question: Calculate the areas of the segments contained between the x-axis and the curve $y = x(x-1)(x-2)$. In order to answer this question, it seems reasonable for us to draw a sketch of the curve, as there is no mention of the ordinates, or values, of x. To make the sketch, we see first that the curve crosses the x-axis when $y = 0$, in other words when $x = 0$, $x = 2$. Next when x is large and positive, we see that y is also large and positive. Finally, when x is large and negative, we see that y is also large and negative. So if we join up these features that we have found on the graph, we can see that the curve looks like the one below.



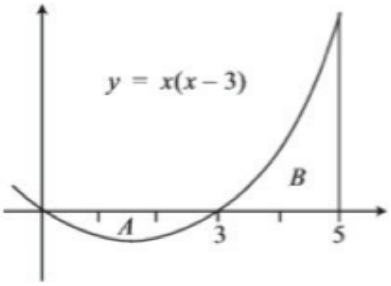
Now the areas required are obviously the area A between $x = 0$ and $x = 1$ and the area B between $x = 1$ and $x = 2$. But there is a marked difference between these two areas in terms of their position. The Area A is above x-axis, whereas the area B is below it. In this case we should calculate each area separately. We know that the area A is given by the integral from $x = 0$ to $x = 1$ of the curve $y = x(x-1)(x-2) = x^3 - 3x^2 + 2x$; thus

$$A = \int_0^1 y\,dx$$
$$= \int_0^1 (x^3 - 3x^2 + 2x)\,dx$$
$$= \left[\frac{x^4}{4} - \frac{3x^2}{3} + \frac{2x^2}{2}\right]_0^1$$
$$= \left[\frac{x^2}{4} - x^3 + x^2\right]_0^1$$
$$= \left[\frac{1}{4} - 1 + 1\right] - \left[\frac{0}{4} - 0 + 0\right]$$
$$= \frac{1}{4}$$

$$B = \int_1^2 y\,dx$$
$$= \int_1^2 (x^2 - 3x^2 + 2x)\,dx$$
$$= \left[\frac{x^4}{4} - \frac{3x^3}{3} + 2\frac{x^2}{2}\right]_1^2$$
$$= [0] - \left[\frac{1}{4} - 1 + 1\right]$$
$$= -\frac{1}{4} \text{ thus area is } \frac{1}{4}.$$

Example

Find the area between the curve $y = x(x-3)$ and the ordinates $x = 0$ and $x = 5$.



Solution:

If we set $y = 0$, we see that $x(x-3) = 0$ and so $x = 0$ and $x = 3$. Thus the curve cuts the $x-$axis at $x = 0$ and at $x = 3$. The term x^2 is positive and so we know that the curve forms U-shape as shown in the diagram.

From the graph, we can calculate area A between the curve, the x-axis and the ordinates $x = 0$ and $x = 3$ first, and that we would expect this integral to give a negative answer because the area is wholly below the x- axis.

Next, we need to calculate area B between the curve, the x-axis and the ordinates $x=3$ and $x=5$:

$$B=\int_3^5 ydx \qquad A=\int_0^3 ydx$$
$$=\int_3^5 (x^2-3x)dx \qquad =\int_0^3 (x^2-3x)dx$$
$$=\left[\frac{x^3}{3}-\frac{3x^2}{2}\right]_3^5 \qquad =\left[\frac{x^3}{3}-\frac{3x^2}{2}\right]_0^3$$
$$=\left[\frac{125}{3}-\frac{3\text{ x }25}{2}\right]-\left[\frac{27}{3}-\frac{3\text{ x }9}{2}\right] \qquad =\left[\frac{27}{3}-\frac{3\text{ x }9}{2}\right]-\left[\frac{0}{3}-\frac{3\text{ x }0}{2}\right]$$
$$=41\frac{2}{3}-37\frac{1}{2}-9+12\frac{1}{2} \qquad =\left[9-\frac{27}{2}\right]-[0]$$
$$=8\frac{2}{3} \qquad =-4\frac{1}{2}$$

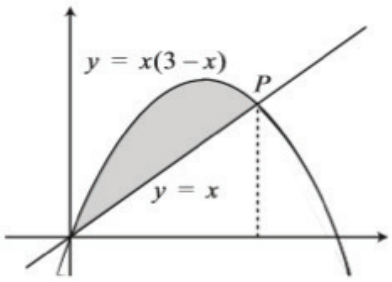
So the total actual area is $4\frac{1}{2}+8\frac{2}{3}=13\frac{1}{6}$ units of area.

3. The area between a curve and a line

Example: Calculate the area of segment cut from the curve $y=x(3-x)$ and by the line $y=x$.

Solution

Sketching both curves on the same axes, we can see by setting $y=0$ that the $y=x(3-x)$cuts the at $x=0$ and $x=3$. Furthermore, the coefficient of x^2 is negative and so we have inverted U – shape curve. The line $y=x$ goes through the origin and meets the curve $y=x(3-x)$at the point P. It is this point that we need to find first of all.



At P the y-ordinates of both curves are equal. Hence;

$$x(3-x)=x$$
$$3x-x^2=x$$
$$2x-x^2=0$$
$$x(2-x)=0$$

So that either, $x=0$, the origin or else $x=2$, the x – coordinates of the point P.
We need to find the shaded area in the diagram. To do this we need the area under the upper curve, the graph of $y=x(3-x)$ between the x- axis and the ordinates $x=0$ and $x=2$. Thus we need to subtract this area with the area under the under the lower curve, the line $y=x$, and between the x- axis and the ordinates $x=0$ and $x=2$. The area is found using this;

$$\int_0^2 [x(3-x)-(x)]dx=\int_0^2 [3x-x^2-x]dx=\int_0^2 [2x-x^2]dx=\left[x^2-\frac{1}{3}x^3\right]_0^2=\frac{4}{3}\text{sq.units}$$

When two different functions enclose or bound a region whose area is to be found, subtract the function below from the top.

4. Area between two curves

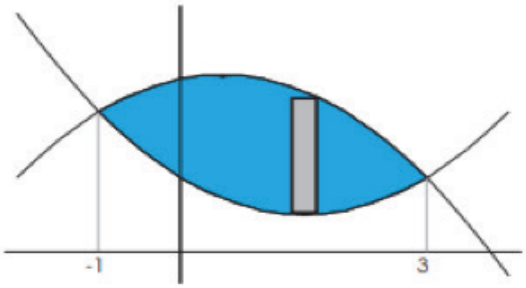
Find the area of the region bounded by the $y=x^2-3x+12$ and $y=18+x-x^2$.
Solution: Find the coordinates of the points of intersection by solving these two simultaneously.

$$y=x^2-3x+12 \text{ and } y=18+x-x^2 \text{ so } x^2-3x+12=18+x-x^2 \Rightarrow x^2-2x-3=0$$
$$\Rightarrow (x-3)(x+1)=0 \Rightarrow x=3 \text{ or } -1.$$

These intercepts are the lower and upper boundaries. We subtract the one below from the top then integrate and evaluate the area.

$$\int_{-1}^{3}[(18+x-x^2)-(x^2-3x+12)]dx = \int_{-1}^{3}[-2x^2+4x+6]dx =$$

$$= \left[-\frac{2}{3}x^3+2x^2+6x\right]_{-1}^{3} = \frac{64}{3}\text{sq.units}$$



Strand 4: Statistics and probability

Content Standard: Students will be able to investigate how to interpret data using methods of exploratory data analysis, develop and evaluate inferences, predictions and arguments that are based on data and understand and apply basic notions of chance and probability.

Units	Benchmark	Topics	Suggested Lesson Titles
Bivariate Statistics	**12.4.4.1** Plot scatter diagrams, draw line of best fit and recognize linear and nonlinear relationship.	Bivariate data on scatter diagrams.	Scatter Diagrams
			Linear and non-linear Relationship
	12.4.4.2 Calculate regression and correlation coefficients, write equation of the regression line, and interpret regression and correlation coefficients in the context of the problem.	Linear Regression	Residual
			Equation of Regression Line
		Correlation	Types of Correlation
			Regression and Correlation Coefficient
	12.4.4.3 Use appropriate technology to aid concept development as a tool for problem solving.	Use of Technology in problem solving	Using a Calculator
			Using an Excel Function

Unit: Bivariate Statistics	Topic : Bivariate data on scatter diagrams

Benchmark	**12.4.4.1** Plot scatter diagrams draw line of best fit and recognize linear and nonlinear relationship.

Learning Objective: By the end of the topic, students will be able to;

- define bivariate data and understand what it means, and
- plot scatter diagram and recognize relationship of bivariate data.



Essential questions:

- What is bivariate data?
- What are some examples of bivariate data?



Key Concepts(ASK-MT)	
Attitudes/Values	Appreciate the concept of bivariate data and be able to describe and explain the relationships between the variables.
Skills	Describing the relationships between variables given bivariate data.
Knowledge	Bivariate data and sketching the data.
Mathematical Thinking	Think about how to analyse bivariate data and explain the results displayed on a graph.

Content Background

1. Bivariate Data

We often want to know if there is a relationship between the items in two different data sets. Examples of these are:

- Is there a relationship between children's ages and their heights?
- Is there a relationship between people's height and weights?
- Is there a relationship between student's marks in an English examination and their marks in mathematics examinations?

In each of the above, two pieces of information are to be collected from each person in the investigation and the two data sets are to be compared. When two pieces of information are collected from each subject in an investigation, then we are concerned with **bivariate data.**

A **scatter graph** or **scatter plot** is a type of display that uses coordinates to display values for two variables for a set of data. That data is displayed as a collection of points, each having the value of one of one variable determining the position of the horizontal coordinates and the value of the other variable determining the position of the vertical coordinate.

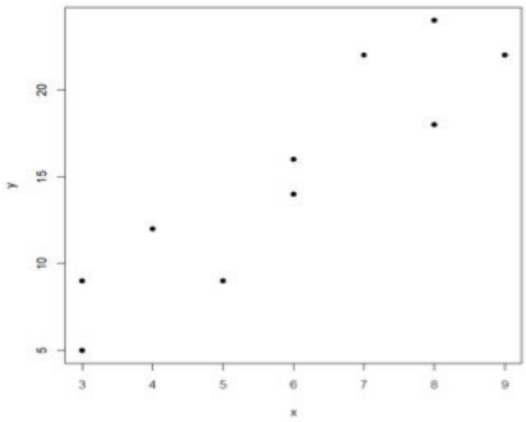
For example, a researcher wishes to investigate whether there is a relationship between the age and the blood pressure of people 50 years or older. To study this, an experiment is conducted using a random sample of ten patients and the following observations are recorded:

Dosage (x)	3	3	4	5	6	6	7	8	8	9
Duration of relief (y)	9	5	12	9	14	16	22	18	24	22

Note: There are two variables in the problem and they are labeled by x and y for our convenience. The common sense suggests that the variable y depends on x and x can be independently selected and controlled by the researcher. However, the variable y cannot be controlled and is dependent on x.

Statisticians are often interested to see whether there is a linear relationship (or linear association) between the two variables x and y: Such observations as collected as pairs on x and y (or $(x;y)$) are called bivariate data.

For the above bivariate data, '$x_1 = 3$' corresponds to '$y_1 = 9$' and there are $n = 10$ pairs. In general, there are n pairs of such bivariate data given by $(x_1,y_1),(x_2,y_2)....(x_n,y_n)$. A **scatter plot** of the above bivariate data is given below.



This scatter plot shows that the points seem to cluster around a straight line. It tells us that there is a possible (approximate) linear relationship between x and y.

Then we want to further investigate: *Is the linear relationship between the variables* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *clear and significant?*

We need to find a measure to investigate the strength of a possible linear relationship between two variables x and y: This measure is known as the correlation coefficient (or Pearson's correlation coefficient) between x and y.

Unit: Bivariate Statistics	Topic : Linear Regression
Benchmark	**12.4.4.2** Calculate regression and correlation coefficients, write equation of the regression line, and interpret regression and correlation coefficients in the context of the problem.



Learning Objective: By the end of the topic, students will be able to;

- understand the need for regression lines in determining the degree to which two variables of interest are correlated,
- find the equation of the line of best fit $(y = mx + c)$ from a data set of two variables of interest,
- extrapolate and interpolate from the regression line equation, $(y = mx + c)$ and make predictions.

Essential questions:



- How can you define 'regression'?
- What causes regression?
- What is a regression line?

Key Concepts(ASK-MT)	
Attitudes/Values	Confidently express the relations between two quantities by way of the equation of the line of best fit.
Skills	Identify the relations between two quantities by way of the equation of the line of best fit.
Knowledge	Linear Regression.
Mathematical Thinking	Think about how to draw a line of best fit and equation of a regression line.

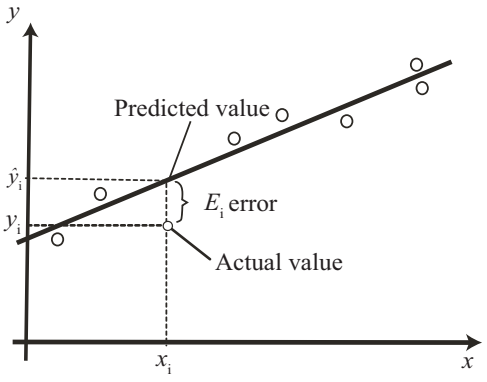
Content Background

Linear Regression

1.Residuals

After verifying that the linear correlation between two variables is significant, next we determine the equation of the line ***(line of best fit)*** that can be used to predict the value of y for a given value of x.

Each data point E_i (error) represents the difference between the observed y-value and the predicted y-value for a given x-value on the line. These differences are called residuals.



2. Regression Line – Least Square method

A regression line, also called a *line* of best fit, is the line for which the sum of the squares of the residuals is a minimum. The equation of a regression line for an independent variable x and a dependent y is $(y = mx + b)$
where $\hat{y}$ is the predicted y – value for the given x – value. The slope m and y- intercept b are given by

$$m = \frac{n\sum xy - (\sum x)(\sum y)}{n\sum x^2 - (\sum x)^2} \text{ and } b = \bar{y} - m\bar{x} = \frac{\sum y}{n} - m\frac{\sum x}{n}$$

where $\bar{y}$ is the mean of the y- values and $\bar{y}$ is the mean of the x-values. The regression line always passes through $(\bar{x},\bar{y})$.

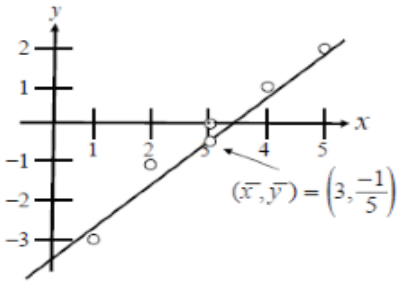
Example 1: Find the equation of the regression line

x	y	xy	x^2	y^2
1	-3	-3	1	9
2	-1	-2	4	1
3	0	0	9	0
4	1	4	16	1
5	2	10	25	4
$\sum x = 15$	$\sum y = -1$	$\sum xy = 9$	$\sum x^2 = 55$	$\sum y^2 = 15$

$$m = \frac{n\sum xy - (\sum x)(\sum y)}{n\sum x^2 - (\sum x)^2} = \frac{5(9)-(15)(-1)}{5(55)-(15)^2} = \frac{60}{50} = 1.2$$

$$b = \hat{y} - m\hat{x} = \frac{-1}{5} - (1.2)\frac{15}{5} = -3.8$$

The regression line is $\hat{y} = 1.2x - 3.8$



Example 2: The following data represents the number of hours 12 different students watched television during the weekend and the scores of each student who took a test the following Monday.

a.) Find the equation of the regression line.

b.) Use the equation to find the expected test score for a student who watches 9 hours of TV.

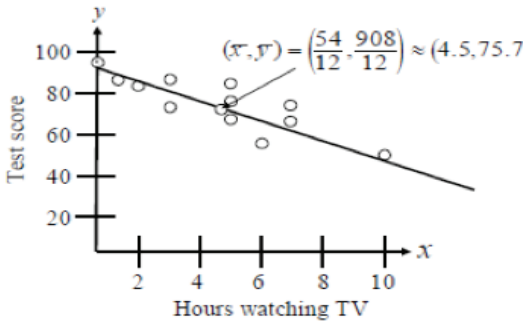
Hours, x	0	1	2	3	3	5	5	6	7	7	10
Test score, y	96	85	82	74	95	68	76	58	65	75	50
xy	0	85	164	222	285	340	380	348	455	525	500
x^2	0	1	4	9	9	25	25	36	49	49	100
y^2	9216	7225	6724	5476	9025	4624	5776	3364	4225	5625	2500

$$\sum x = 54,\ \sum y = 908,\quad \sum xy = 3724, \sum x^2 = 332, \sum y^2 = 7083$$

$$m = \frac{n\sum xy - (\sum x)(\sum y)}{n\sum x^2 - (\sum x)^2} = \frac{12(3724)-(54)(908)}{12(332)-(54)^2} \approx -4.067$$

$$b = y - m\hat{x} = \frac{908}{12} - (-4.067)\frac{54}{12} \approx 93.97$$

$$\hat{y} = -4.07x + 93.97$$



Unit: Bivariate Statistics	Topic: Correlation
Benchmark	**12.4.4.2** Calculate regression and correlation coefficients, write equation of the regression line, and interpret regression and correlation coefficients in the context of the problem.

Learning Objective: By the end of the topic, students will be able to;



- understand what a correlation coefficient is and interpret it
- calculate Correlation Coefficient (r) from a set of data.

Essential questions:



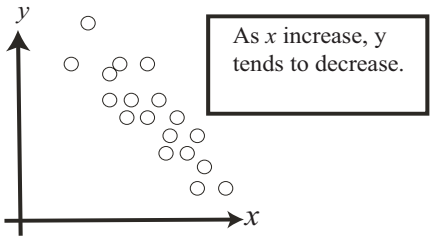
- What is correlation?
- What is the significance of studying correlation?
- What is the meaning to the values of r?
- What is q correlation?

Key Concepts(ASK-MT)	
Attitudes/Values	Express confidence in talking about correlations between quantities and explain the relationship between the variables.
Skills	Identify correlations between quantities and explain the relationship between the variables.
Knowledge	Correlation
Mathematical Thinking	Think about how to identify correlations between quantities and explain the relationship between the variables.

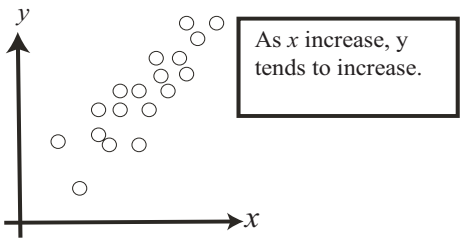
Content Background

1. Types of Correlation

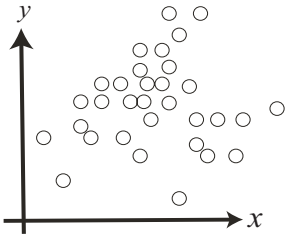
Once the intercept and slope have been estimated using least squares, various indices are studied to determine the reliability of these estimates. One of the most popular of these reliability indices is the correlation coefficient. The correlation coefficient, or simply the correlation, is an index that ranges from -1 to 1. When the value is near zero, there is no linear relationship. As the correlation gets closer to plus or minus one, the relationship is stronger. A value of one (or negative one) indicates a perfect linear relationship between two variables. Actually, the strict interpretation of the correlation is different from that given in the last paragraph. The correlation is a parameter of the bivariate normal distribution. This distribution is used to describe the association between two variables. This association does not include a cause and effect statement. That is, the variables are not labeled as dependent and independent. One does not depend on the other. Rather, they are considered as two random variables that seem to vary together. The important point is that in linear regression, y is assumed to be a random variable and x is assumed to be a fixed variable. In correlation analysis, both y and x are assumed to be random variables.



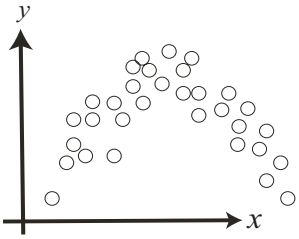
Negative Linear Correlation



Positive Linear Correlation



No Correlation



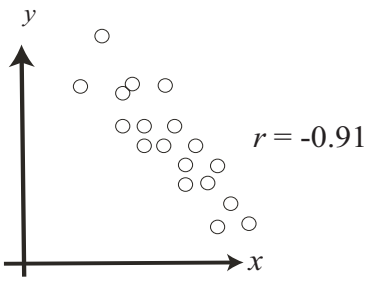
Nonlinear Correlation

Correlation Coefficient

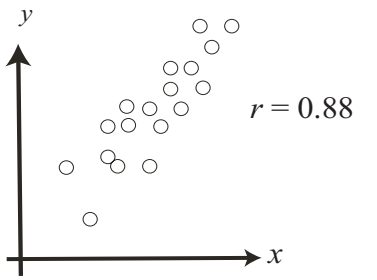
The correlation coefficient is a measure of the strength and the direction of a linear relationship between two variables. The symbol r represents the sample correlation coefficient. The formula for r is

$$r = \frac{n\sum xy - (\sum x)(\sum y)}{\sqrt{n\sum x^2 - (\sum x)^2}\sqrt{n\sum y^2 - (\sum y)^2}}$$

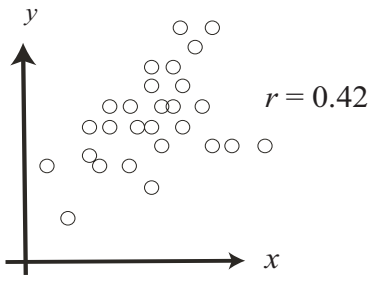
The range of the correlation coefficient r is –1 to 1. If x and y have a strong positive linear correlation, r is close to 1. If x and y have a strong negative linear correlation, r is close to –1. If there is no linear correlation or a weak linear correlation, r is close to 0. r is a measure of the linear (straight-line) association between two variables.



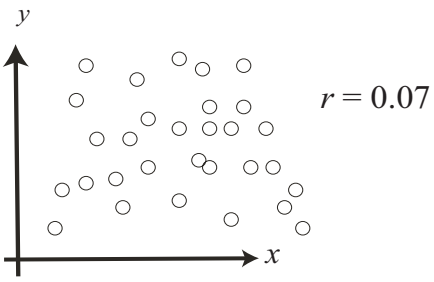
Strong negative correlation



Strong positive correlation



Weak positive correlation



Nonlinear Correlation

Example 1: Calculate the correlation coefficient r for the following data.

x	y	xy	x^2	y^2
1	-3	-3	1	9
2	-1	-2	4	1
3	0	0	9	0
4	1	4	16	1
5	2	10	25	4
$\sum x = 15$	$\sum y = -1$	$\sum xy = 9$	$\sum x^2 = 55$	$\sum y^2 = 15$

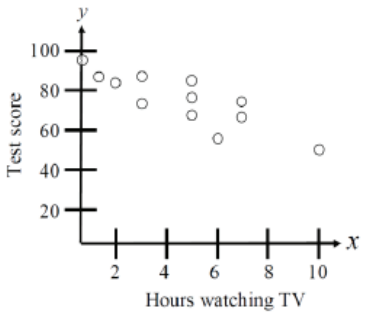
$$r = \frac{n\sum xy - (\sum x)(\sum y)}{\sqrt{n\sum x^2 - (\sum x)^2}\sqrt{n\sum y^2 - (\sum y)^2}} = \frac{5(9) - (15)(-1)}{\sqrt{5(55) - 15^2}\sqrt{5(15) - (-1)^2}}$$

$$= \frac{60}{\sqrt{50}\sqrt{74}} \approx 0.986$$

There is a strong positive linear correlation between x and y.

Example 2: The following data represents the number of hours 12 different students watched television during the weekend and the scores of each student who took a test the following Monday.
a) Display the scatter plot.
b) Calculate the correlation coefficient r.

Hours (x)	0	1	2	3	3	5	5	5	6	7	7	10
Test score (y)	96	85	82	74	95	68	76	84	58	65	75	50



Hours (x)	0	1	2	3	3	5	5	5	6	7	7	10
Test score (y)	96	85	82	74	95	68	76	84	58	65	75	50
xy	0	85	164	222	285	340	380	420	348	455	525	500
x^2	0	1	4	9	9	25	25	25	36	49	49	100
y^2	9216	7225	6724	5476	9025	4624	5776	7056	3364	4225	5625	2500

$\sum x = 54$, $\sum y = 908$, $\sum xy = 3724$, $\sum x^2 = 332$, $\sum y^2 = 70836$

$$r = \frac{n\sum xy - (\sum x)(\sum y)}{\sqrt{n\sum x^2 - (\sum x)^2}\sqrt{n\sum y^2 - (\sum y)^2}} = \frac{12(3724) - (54)(908)}{\sqrt{12(332) - 54^2}\sqrt{12(70836) - (908)^2}} \approx -0.831$$

There is a strong negative linear correlation. As the number of hours spent watching TV increases, the test scores tend to decrease.

Unit: Bivariate Statistics	Topic : Use of Technology in Problem solving
Benchmark	**12.4.4.3** Use appropriate technology to aid concept development as a tool for problem solving.

Learning Objective: By the end of the topic, students will be able to;



- use appropriate technologies to do basic statistical calculations such as mean, and variance, standard deviation, correlation coefficient, covariance etc.

Essential questions:



- what statistical application or software is used in calculating the mean, variance, standard deviation, correlation coefficient, covariance etc either from the smartphone or laptop?
- How efficient and effective is the statistical application or software?

Key Concepts(ASK-MT)	
Attitudes/Values	Be appreciative to apply technology to calculate statistical problems.
Skills	Apply technology to calculate statistical problems.
Knowledge	Using technology to calculate the basic statistical calculations.
Mathematical Thinking	Think about how to use technology to evaluate satistical calculations.

Content Background

This topic is for students to make use of available technology to aid the concept development statistical calculation such as calculation of linear regression and correlation co-efficient of a set of bivariate numerical data. Calculators and MS Excel are examples of technologies which can be used to aid concept development.

Examples

1. Using the Calculator

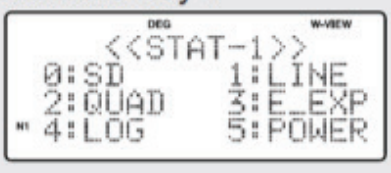
We can use a calculator to find equation of a line of linear regression. When a series of bivariate data has been entered correctly, then the calculator can be used to find the values of a and b, to give the equation of the line of regression from which we can make estimates of the variable y, if we know a value for the other variable x.

Set the calculator in Statistics mode.

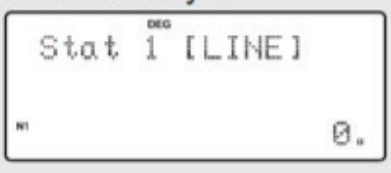
MODE key



Numeric key 1



Numeric key 1



The calculator is now set for bivariate data entry; each pair p, q being input as p [(x, y) key] q [DATA key]. For example, if the bivariate pair 0, 2 were the first item of a set of data, they would appear on the calculator as:

For the following table, enter the bivariate data shown. The calculator screens are shown for the whole process

p	0	1	2	3	4
q	2	3	5	4	6

Having entered the data, the values of the **a** and **b** in the equation of the line of regression of y on x, $y = \mathbf{a} + \mathbf{b}\,x$, can be found using the calculator. The ALPHA key identifies **a** and **b** as extra functions on the bracket keys



ALPHA **a** =



ALPHA **b** =

The line of regression of y on x is therefore $y = 2.2 + 0.9\,x$

Estimates of the value of y can be found by substituting given values of x in the equation.



2. Using an Excel function

We can calculate correlation coefficient for the following data using an excel function.

x	1.0	2.3	3.1	4.8	5.6	6.3
y	2.6	2.8	3.1	4.7	5.1	5.3

The Correlation Coefficient is calculated according to the following formula:

$$r = \frac{n\sum xy - \sum x \sum y}{\sqrt{\left[n\sum x^2 - \left(\sum x\right)^2\right]\left[n\sum y^2 - \left(\sum y\right)^2\right]}}$$

It may appear that the above equations are quite complicated, however upon inspection; we see that their components are nothing more than simple algebraic manipulations of the raw data. We can expand our spread sheet to include these components and use inbuilt function to do our calculations.

A11 = =COUNT(B3:B8)

	A	B	C	D	E	F	G	H	I	J
1										
2		x	y	xy	x^2	y^2				
3		1.0	2.6	2.6	1.0	6.8				
4		2.3	2.8	6.44	5.3	7.8				
5		3.1	3.1	9.61	9.6	9.6				
6		4.8	4.7	22.56	23.0	22.09				
7		5.6	5.1	28.56	31.4	26.0				
8		6.3	5.3	33.39	39.7	28.1				
9										
10	n	Σ x	Σ y	Σ (xy)	Σ (x^2)	Σ (y^2)				
11	6	23.1	23.6	103.16	110.0	100.4				
12										
13		$(\Sigma x)^2$	$(\Sigma y)^2$							
14		533.61	556.96							
15										
16	slope, m =	0.5842		=(A11*D11-B11*C11)/(A11*E11-B14)						
17	y-int, b =	1.6842		=(C11-C16*B11)/A11						
18	r =	0.9741		=(A11*D11-B11*C11)/SQRT((A11*E11-B14)*(A11*F11-C14))						

Standards-Based Lesson Planning

What are Standards-Based Lessons?

In a Standards-Based Lesson, the most important or key distinction is that, a student is expected to meet a defined standard for proficiency. When planning a lesson, the teacher ensures that the content and the methods of teaching the content enable students to learn both the skills and the concepts defined in the standard for that grade level and to demonstrate evidence of their learning.

Planning lessons that are built on standards and creating aligned assessments that measure student progress towards standards is the first step teacher must take to help their students reach success. A lesson plan is a step-by-step guide that provides a structure for an essential learning.

When panning a standards-based lesson, teacher instructions are very crucial for your lessons. How teachers instruct the students is what really points out an innovative teacher to an ordinary teacher. Teacher must engage and prepare motivating instructional activities that will provide the students with opportunities to demonstrate the benchmarks. For instance, teacher should at least identify 3-5 teaching strategies in a lesson; teacher lectures, ask questions, put students into groups for discussion and role play what was discussed.

Why is Standards-Based Lesson Planning Important?

There are many important benefits of having a clear and organized set of lesson plans. Good planning allows for more effective teaching and learning. The lesson plan is a guide and map for organizing the materials and the teacher for the purpose of helping the students achieve the standards. Lesson plans also provide a record that allows good, reflective teachers to go back, analyse their own teaching (what went well, what didn't), and then improve on it in the future.

Standards-based lesson planning is vital because the content standards and benchmarks must be comparable, rigorous, measurable and of course evidence based and be applicable in real life that we expect students to achieve. Therefore, teachers must plan effective lessons to teach students to meet these standards. As schools implement new standards, there will be much more evidence that teachers will use to support student learning to help them reach the highest levels of cognitive complexity. That is, students will be developing high-level cognitive skills.

Components of a Standards-Based Lesson Plan

An effective lesson plan has three basic components;

- aims and objectives of the course;
- teaching and learning activities;
- assessments to check student understanding of the topic.

Effective teaching demonstrates deep subject knowledge, including key concepts, current and relevant research, methodologies, tools and techniques, and meaningful applications.

Planning for under-achievers NORMA

Who are underachieving students?
Under achievers are students who fail or do not perform as expected. Underachievement may be caused by emotions (low self-esteem) and the environment (cultural influences, unsupportive family)

How can we help underachievement?

Underachievement varies between students. Not all students are in the same category of underachievement.
Given below a suggested strategies teachers may adopt to assist underachievers in the classroom.

• Examine the Problem Individually
It is important that underachieving students are addressed individually by focusing on the student's strengths.

• Create a Teacher-Parent Collaboration
Teachers and parents need to work together and pool their information and experience regarding the child. Teachers and parents begin by asking questions such as;

- In what areas has the child shown exceptional ability?
- What are the child's preferred learning styles?
- What insights do parents and teachers have about the child's strengths and problem areas?

• Help student to plan every activity in the classroom
• Help students set realistic expectations
• Encourage and promote the student's interests and passions.
• Help children set short and long-term academic goals
• Talk with them about possible goals.
• Ensure that all students are challenged (but not frustrated) by classroom activities
• Always reinforce students

Example of Standards-Based Lesson Planning

The following sample lesson can help teachers to plan effective lessons. Teachers are encouraged to study the layout of the different components of these lessons and follow this design in their preparation and teaching of each lesson. Planning a good lesson helps the teacher in maintaining a standard teaching pattern which should not deviate students learning of the concept from the topic.

Sample Lesson Plan

TOPIC: Introduction to Complex Numbers

Lesson Topic: Standard and Derived Units
Grade: 12-Advanced Mathematics
Length of Lesson: 40 minutes

STRAND 1: PATTERNS AND ALGEBRA - Students will be able to represent numbers in various situations and forms, develop fluency in calculations through operations, use base ten as key for extending numbers and operations, and apply numbers in practical situations to develop number sense.

Benchmark: 12.1.1.1 Use the complex number system, the notation for complex numbers, and the definition of "*i*" to solve problems in standard form.

Lesson Title: Then Need for Complex Numbers

Lesson Objective: By the end of the lesson, students should be able to define and explain significance of complex numbers.

Materials: Activity handouts, charts and white/blackboard and chalk

Key Concepts (ASK-MT)	
Attitudes/Values	Acknowledging that there is complex number existing which is useful in some mathematical computation making math fun and exciting.
Skills	Representing complex numbers by using the appropriate notations
Knowledge	• Origination of Complex Numbers and its purpose in the number system. • Know that real numbers are part of Complex Numbers, the notations, the imaginary (i), conjugate and square roots of complex numbers.
Mathematical Thinking	Think of how *i* was created to solve problems such as "$\sqrt{-39}$" and "$x^2 + 1 = 0$"

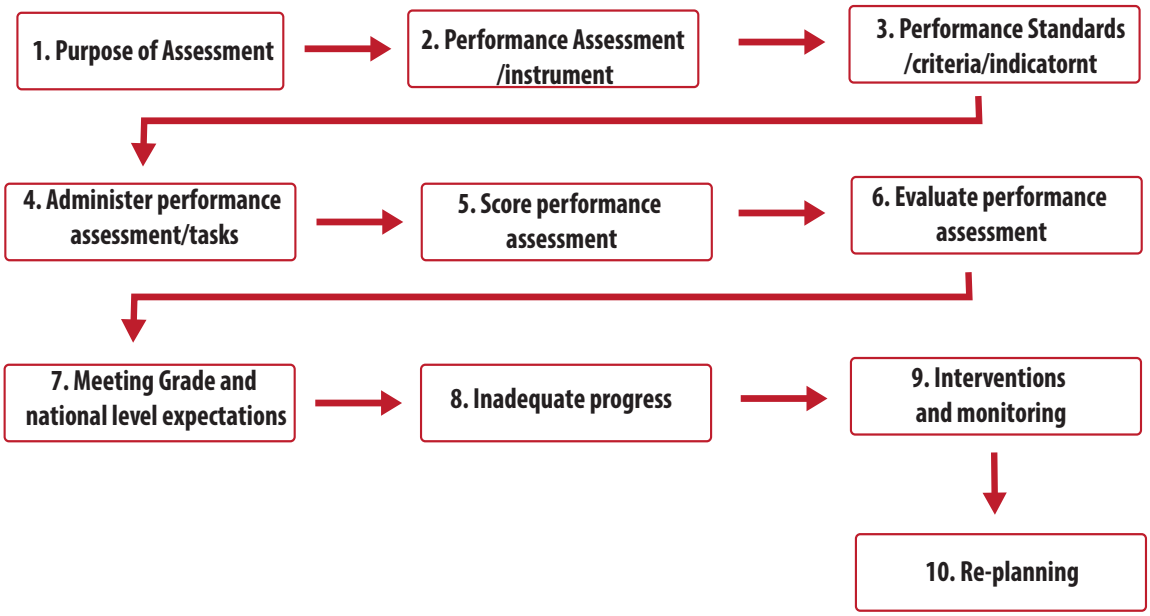
Lesson Procedure

Teacher Activities	Student Activities
Introduction	
Ask students solve the quadratic equation $x^2 + 1 = 0$ Let students think about the problem. Ask Essential Questions (relating to current lesson topic) Can you find a real solution to this equation? Can you think of some other ways to make the quadratic equation above have a real root?	• Listen to the teacher. • Think about how to solve $x^2 + 1 = 0$ • Responds to teacher's questions.
Body	
Modelling	
• Explain the need for complex numbers using quadratic equation and formula to determine real solutions. • Ask students to demonstrate graphical view point for quadratic equation when $b^2 - 4ac < 0$. • Define complex number and write its notation.	• Listen attentively to teacher and recognise the need for complex nubmers. • Think about and draw sketch of quadratic equation when $b^2 - 4ac < 0$ and recognise that the curve lie entirely above ro below the x-axis. • Evaluate the definition for complex number and its notation.
Guided Practice	
• Provide 2 or 3 examples and ask students to explain the reason for impossible solutions.	• Listen to teacher's instruction • Explain the reason for imposible solutions and appreciate the use of complex numbers to represent them.
Independent Practice	
• Provide 2 or 3 problems for students to solve. • Allow for independent or group discussion to solve the problems.	• Attend to the given questions individually or through group discussion.
Conclusion	
• Ask students to present their solutions on the board for class discussion. • Make corrections where necessary and emphasise on key points.	• Present their solutions on the board for class discussions. • Make corrections when necessary and listen to teacher's key points.
Assessment or Lesson Evaluation	
Students can; • Define complex numbers. • Appreciate the need for complex number. • Have real solution to quadratics equation with $b^2 - 4ac < 0$.	

Assessment, Monitoring and Reporting

What is Standards-Based Assessment (SBA)?

Standards-Based Assessment is an on-going and a systematic process of **assessing, evaluating, reporting** and **monitoring** students' performance and progression towards meeting grade and national level expectations. It is the measurement of students' proficiency on a learning objective or a specific component of a content standard and progression towards the attainment of a benchmark and content standard.



Purpose of Standards-Based Assessment

Standards-Based Assessment (SBA) serves different purposes. These include instruction and learning purposes. The primary purpose of SBA is to improve student learning so that all students can attain the expected level of proficiency or quality of learning.

Enabling purposes of SBA is to:

- measure students' proficiency on well-defined content standards, benchmarks and learning objectives
- ascertain students' attainment or progress towards the attainment of specific component of a content standard
- ascertain what each student knows and can do and what each student needs to learn to reach the expected level of proficiency
- enable teachers to make informed decisions and plans about how and what they would do to assist weak students to make adequate progress towards meeting the expected level of proficiency
- enable students to know what they can do and help them to develop and implement strategies to improve their learning and proficiency level
- communicate to parents, guardians, and relevant stakeholders the performance and progress towards the attainment of content standards or its components
- compare students' performances and the performances of other students

Principles of Standards-Based Assessment

The principle of SBA is for assessment to be;

- emphasising on tasks that should encourage deeper learning
- be an integral component of a course, unit or topic and not something to add on afterward
- a good assessment requires clarity of purpose, goals, standards and criteria
- of practices that should use a range of measures allowing students to demonstrate what they know and can do
- based on an understanding of how students learn
- of practices that promote deeper understanding of learning processes by developing students' capacity for self-assessment
- improving performance that involves feedback and reflection
- on-going rather than episodic
- given the required attention to outcomes and processes
- be closely aligned and linked to learning objectives, benchmarks and content standards

Standards-Based Assessment Types

In standards-Based Assessment, there are three broad assessments types.

1. Formative Assessment

Formative assessment includes 'assessment *for* and *as'* and is conducted during the teaching and learning of activities of a topic.

Purposes of assessment for Learning

- On-going assessment that allows teachers to monitor students on a day-to-day basis.
- Provide continuous feedback and evidence to the teachers that should enable them to identify gaps and issues with their teaching, and improve their classroom teaching practice.
- Helps students to continuously evaluate, reflect on, and improve their learning

Purposes of assessment as Learning

- Occurs when students reflect on and monitor their progress to inform their future learning goals
- Helps students to continuously evaluate, reflect, and improve their own learning
- Helps students to understand the purpose of their learning and clarify learning goals

2. Summative Assessment

Summative assessment focuses on 'assessment *of learning'* and is conducted after or at the conclusion of teaching and learning of activities or a topic.

Purposes of assessment of Learning

- Help teachers to determine what each student has achieved and how much progress he/she has made towards meeting national and grade-level expectations
- Help teachers to determine what each student has achieved at the end of a learning sequence or a unit.
- Enable teachers to ascertain each student's development against the unit or topic objectives and to set future directions for learning.
- Help students to evaluate, reflect on, and prepare for next stage of learning

3. Authentic Assessment

- Is performed in a real life context that approximates as much as possible, the use of a skill or concept in the real world.
- Is based on the development of a meaningful product, performance or process
- Students develop and demonstrate the application of their knowledge, skills, values and attitudes in real life situations which promote and support the development of deeper levels of understanding.

Authentic Assessment Criteria

Authentic assessment refers to assessment that:

- Looks at students actively engaged in completing a task that represents the achievement of a learning objective or standard
- Takes place in real life situations
- Asks students to apply their knowledge, skills, values and attitudes in real life situations
- Students are given the criteria against which they are being assessed

Performance Assessment

Performance assessment is a form of testing that requires students to perform a task rather than select an answer from a ready-made list. For example, a student may be asked to explain historical events, generate scientific hypotheses, solve math problems, converse in a foreign language, or conduct research on an assigned topic. Teachers, then judge the quality of the student's work based on an agreed-upon set of criteria. It is an assessment which requires students to demonstrate that they have mastered specific skills and competencies by performing or producing something.

Types of performance assessment;

i. Products

This refers to concrete tangible items that students create through either the visual, written or auditory media such as;

- Creating a health/physical activity poster
- Video a class game or performance and write a broadcast commentary

- Write a speech to be given at a school council meeting advocating for increased time for health and physical education in the curriculum
- Write the skill cues for a series of skill photo's
- Create a brochure to be handed out to parents during education week
- Develop an interview for a favourite sports person
- Write a review of a dance performance
- Essays
- Projects

ii. Process Focused Tasks

It shows the thinking processes and learning strategies students use as they work such as;

- Survival scenarios
- Problem solving initiative/adventure/ activities
- Decision making such as scenario's related to health issues
- Event tasks such as creating a game, choreographing a dance/gymnastics routine, creating an obstacle course
- Game play analysis
- Peer assessment of skills or performances
- Self-assessment activities
- Goal setting, deciding a strategy and monitoring progress towards achievement

iii. Portfolio

This refers to a collection of student work and additional information gathered over a period of time that demonstrates learning progress.

iv. Performances

It deals with observable affective or psycho-motor behaviours put into action such as;

- Skills check during game play
- Role plays
- Officiating a game
- Debates
- Performing dance/gymnastics routines
- Teaching a skill/game/dance to peers

Performance Standards

Performance Standards are concrete statements of how well students must learn what is set out in the content standards, often called the "be able to do" of "what students should know and be able to do." Performance standards are the indicators of quality that specify how competent a students' demonstration or performance must be. They include explanations of how well students must demonstrate the content, explaining how good is good enough.

Performance standards:

- measure students' performance and proficiency (using performance indicators) in the use of a specific knowledge, skill, value, or attitude in real life or related situations
- provide the basis (performance indicators) for evaluating, reporting and monitoring students' level of proficiency in use of a specific knowledge, skills, value, or attitude

- are used to plan for individual instruction to help students not yet meeting expectations (desired level of mastery and proficiency) to make adequate progress towards the full attainment of benchmarks and content standards
- are used as the basis for measuring students' progress towards meeting grade-level benchmarks and content standards

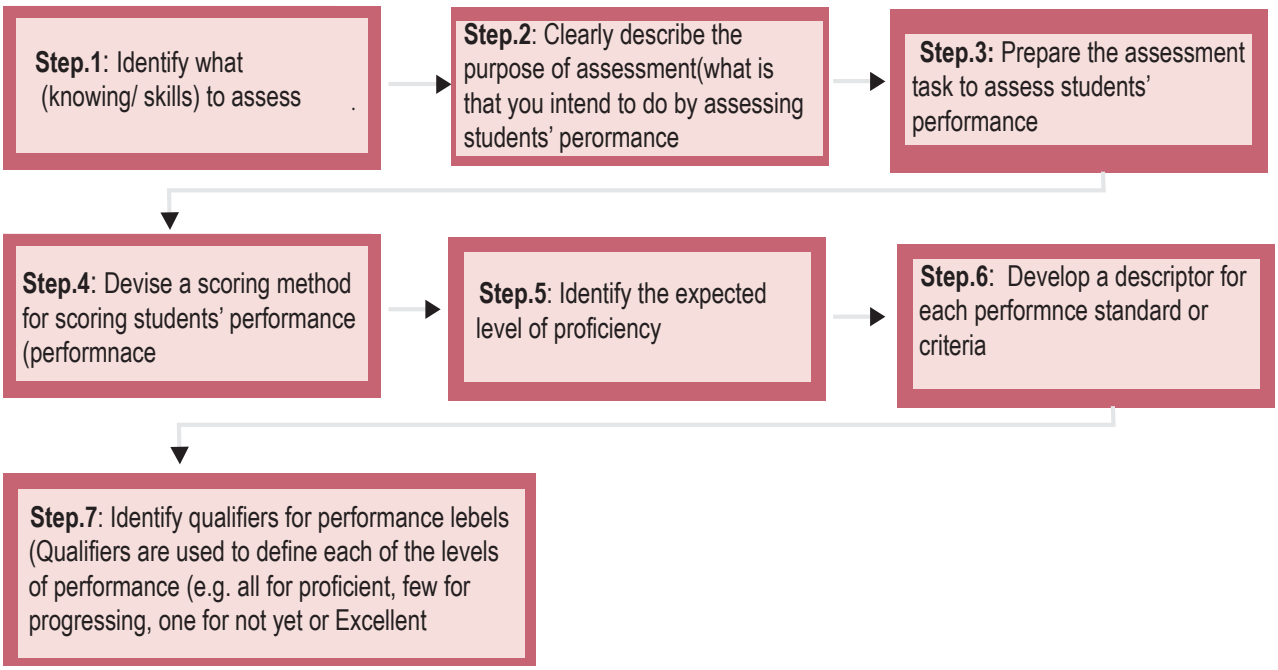
Assessment Strategies

It is important for teachers to know that, assessment is administered in different ways. Assessment does not mean a test only. There are many different ways to find out about student's strengths and weaknesses. Relying on only one method of assessing will not reflect student's achievement.

Provided in the appendices is a list of suggested strategies you can use to assess student's performances. These strategies are applicable in all the standards-based assessment types.

How to Develop effective SBA tools

Teachers are required to use the steps outlined below when planning assessment. These steps will guide you to develop effective assessments to improve student's learning as well as evaluating their progress towards meeting national and grade –level expectations.



Sample of Assessment Types

Formative

Sample 1
STRAND 1: Number and Operation
Content Standard: Students will be able to interpret various types of patterns and functional relationships, use symbolic forms to represent, model, and analyze mathematical situations and collect, organize, and represent data to answer questions.

Topic: Introduction to Complex Numbers
Benchmark: 12.1.1.1 Use the complex number system, the notation for complex numbers, and the definition of "i" to solve problems in standard form.

Lesson Topic: The Need for Complex Numbers

Objective: By the end of the lesson, students should be able to;
- define and explain significance of complex numbers.

VASK-MT	
Attitudes /Values	Acknowledging that there is complex number existing which is useful in some mathematical computation making math fun and exciting.
Skills	Representing complex numbers by using the appropriate notations
Knowledge	• Origination of Complex Numbers and its purpose in the number system. • Know that real numbers are part of Complex Numbers, the notations, the imaginary (i), conjugate and square roots of complex numbers.
Mathematical Thinking	Think of how i was created to solve problems such as "$\sqrt{-39}$" and "$x^2 + 1 = 0$"

What is to be assessed? - (KSAVs)

Define and represent complex numbers and appreciate its purpose to make mathematics fun and exciting.

Purpose of the assessment

To measure students' proficiency on the achievement of the benchmark and learning objectives.

Expected level of proficiency

Define and explain significance of complex numbers to solve problems such as "$\sqrt{-39}$" and "$x^2 + 1 = 0$"

Performance Task

Define complex numbers and solve problems such as "$\sqrt{-39}$" and "$x^2 + 1 = 0$"

Assessment Strategy

This assessment can be conducted in one lesson as an assessed lesson exercise.

Assessment Scoring

Rubrics must be developed to articulate the real proficiency of thechild. This is an analytical rubrics used to assess the child's learning through the assessment tool a lesson exercise.

Performance standards/ Criteria	A	B	C	D	Score
	Advance 10	Proficient 9 - 5	Progressing 3 - 4	Not Yet 2	__/10 Marks
Define and explain significance of complex numbers to solve problems such as "$\sqrt{-39}$" and "$x^2 + 1 = 0$" **10 marks**	Define complex numbers and solve problems such as "$\sqrt{-39}$" and "$x^2 + 1 = 0$" with calculation steps.	Correct definition of complex numbers and correctly answered all the related questions.	Satisfactory definition of complex numbers and correctly answered some the related questions.	Poor definition of the complex numbers and answered only one of the related questions.	

Summative

Sample 2

STRAND 4: Statistics and Probability

Content Standard: Students will be able to investigate how to interpret data using methods of exploratory data analysis, develop and evaluate inferences, predictions and arguments that are based on data and understand and apply basic notions of chance and probability.

Unit: Bivariate Statistics

Benchmark 12.4.4.1- 12.4.4.3: (REFER TO THE BENCHMARKS IN UNIT: Bivariate Statistics OF STRAND 4)

Topics 1- 4: (REFER TO THE TOPICS IN UNIT: Bivariate Statistics of strand 4)

Lesson topics: (refer to the lesson topics in Bivariate Statistics of strand 4)

Instructional Objective (s): (Refer to Unit: Bivariate Statistics of Strand 4)

VASK-MT	
Values/ Attitudes	Appreciate the usefulness of Bivariate Statistics to solve problems.
Skills	Analyse Bivariate data
Knowledge	Bivariate Statistics calculations
Mathematical Thinking	Think about how to analyse bivariate data.

Performance standards/Criteria	A	B	C	D	Score
	Advance 20	Proficient 13-19	Progressing 6-12	Not Yet 2-5	__/20 Marks
(10 marks) Criteria/ Constraints	Assignment was completed with all constraints and criteria met or exceeded. Reflects attention to detail and quality.	Assignment was completed with some of the constraints and criteria met. Reflects some attention to detail, but quality is minimal.	Assignment was completed with a few of the constraints and criteria met. Reflects minimal effort and lacks detail or quality.	Assignment was not completed and does not reflect the adherence to the constraints or criteria.	
(10 marks) Presentation of data, scatter plot and regression line.	Correct sketch of the scatter plot and regression line and represented all information correctly and answered all the related questions with clear calculation steps	Correct sketch of the scatter plot and regression line and represented all information correctly and answered all the related questions.	Satisfactory sketch of the scatter plot and regression line and represented most information correctly and answered some of the related questions.	Shows poor knowledge of the scatter plot and regression line representations	
Analysis. (3 marks)	Student carefully analyzed the information collected and drew appropriate and inventive conclusions supported by the evidence.	Student shows good effort in analyzing the evidence collected.	Student conclusions could have be supported by stronger evidence. Level of analysis could have been deeper	Student conclusions simply involved restating information. Conclusions were not supported by evidence.	
Time Management	Assignment completed and turned in on time. Student worked diligently when assignment time was available. Student was on task most of the time.	Assignment was completed, but had notable errors. Student utilized assignment time somewhat efficiently, but spent time socializing. Student was on task 70% - 80% of the time.	Assignment was not turned in on time and/or complete. The student was on task less than 60% of the time.	Assignment was not turned in on time and was not completed. Student wasted Assignment time and at times was disruptive to others.	

Authentic Assessment
Sample 3
Strand 2: Geometry, Measurement and Transformation
Content Standard: Students will be able to comprehend the meaning and significance of geometry, measurements and spatial relationship including units and system of measurement and develop and use techniques, tools, and formulas for measuring the properties of objects and relationships among the properties and use transformations and symmetry to analyze
Unit : Trigonometry

Benchmark 12.2.2.1 – 12.2.2.9 (Refer to the Benchmarks In Unit: Trigonometry, Strand 2)
Topics: (Refer to the topics In Unit: Trigonometry, Strand 2)

Lesson topics: (Refer to the Topics In Unit: Trigonometry, Strand 2)

Instructional Objective (s): (Refer to the topics In Unit: Trigonometry, Strand 2)

Key Concepts(ASK-MT)	
Attitudes/Values	Appreciate how real life problems can be described using trigonometric functions and displaying it on a graph.
Skills	Relate and represent real life problems to trigonometric functions
Knowledge	Real world phenomena and the function $y = A\sin(Bx + C) + D$
Mathematical Thinking	Analyzing real world phenomena relating to the trigonometric function $y = A\sin(Bx + C) + D$

What is to be assessed? - (KSAVs)

The unit (Unit) "Trigonometry"

Purpose of the assessment

To measure students proficiency on the achievement of the benchmarks and learning objectives in this unit. This assessment is to be conducted after teaching this unit.

Expected level of proficiency

All students are expected to;

- Analyse and represent real world phenomena represented by the trigonometric function in the form of $y = A\sin(Bx + C) + D$

Performance Task

- Students will do a Real World Application Project (Sinusoidal Modeling) out of 30 marks. You can use other assessment tools (assignment, projects, etc.) assess student's proficiency on these benchmarks.
- Task: Students will be given three weeks to complete this project. They are to;
- Investigate any real-world phenomenon that can be modelled by a sinusoid.
- Research and collect data, develop a predictive model, graph it, and present it to the class using a visual presentation platform like, iMovie, Powerpoint, etc.
- Choose any topic, and the freedom to be as creative and outside-the-box with this project as they wish.
- Note: Students should now understand that any variable that is cyclical, harmonic, oscillating, or periodic in nature can be modelled graphically by a sine or cosine wave. There are countless applications of sinusoid modelling in real life. Some of these applications include:
- Changes in Temperature over time
- Hours of daylight over time
- Population growth/decay over time
- Ocean wave heights (high and low tides) over time
- Sound waves
- Electrical currents
- Ferris wheels and roller coasters
- Tsunamis and tidal waves
- Earthquakes
- Wheels and Swings
- Task Details: Students are to;
- Collect no less than 12 real-world data points that can be modeled sinusoidally. The more data, the better! Providing 24 or more data points will earn maximum

- points on the rubric.
- A predictive model of the format y = Asin (Bx + C)+ D or y = Acos (Bx + C) + D for the data must be developed using the techniques they learned in class. This model could be used to predict outcomes into the future.
- A neatly organized graph of the original data and a graph of their developed model must both be embedded in their presentation. To produce these graphs they may use their graphing calculators, the Desmos app, or they may draw the graphs themselves. Pictures or screenshots of their graphs may be used. The objective is to compare the two graphs side-by-side or on top of one another, so they could see how accurate and predictive their model is.
- A reflection must be submitted with your project (1-2 paragraphs). This reflection should be neatly and logically written/typed with no grammatical errors, and should summarize their experience in doing this project. What did they learn? What did they enjoy or dislike? What would they change? How well did they work with their partner? Etc.

Criteria	Model/Exemplar (4 points	Proficient (3 points)	Developing (2 points)	Beginning (1 point)	Score
Data Collection	Data is authentic, appropriately labeled and clearly presented in an X-Y table. Contains 24 or more measurements.	Data is authentic, appropriately labeled and clearly presented in an X-Y table. Contains 13-23 measurements.	Data is authentic, appropriately labeled and clearly presented in an X-Y table. Contains 12 measurements.	Data is incorrectly labeled, not presented in an X-Y table, and contains less than 12 measurements.	
Mathematical Calculations/Model Development	All calculations are very clear, organized, and neatly completed with no inaccuracies.	All calculations are clear, organized, and neatly completed with 1-2 inaccuracies.	Most calculations are clear, organized, and neatly completed with 3-4 inaccuracies.	Calculations are unclear and disorganized and 5 or more inaccuracies may be present.	
Graphs	All graphs are neatly produced, axes are appropriately scaled and labeled, data points are accurately plotted, colorful, and smooth curves are drawn.	All graphs are neatly produced, axes are appropriately scaled and labeled, data points are accurately plotted, colorful, and smooth curves are drawn.	All graphs are not neatly produced, axes are not appropriately scaled and labeled, data points are not accurately plotted, and smooth curves are not drawn.	All graphs are not neatly produced, axes are not appropriately scaled and labeled, data points are not accurately plotted, and smooth curves are not drawn	
Visual Presentation	The presentation is clear, colorful, creative and entertaining, shows a great deal of editing and audio/visual effects, keeps the audience fully engaged, fully utilizes available technology, and lasts 5-10 minutes.	The presentation is clear and colorful, shows some editing and audio/visual effects, keeps the audience mostly engaged, and fully utilizes technology, and lasts 3-5 minutes	The presentation is bland and basic, does not show editing or effects, keeps the audience moderately engaged, and does not fully utilize technology, and lasts 1-2 minutes	The presentation is erratic and poorly produced, lacks effort, does not show any editing or effects, the audience is not engaged, and does not utilize technology, and lasts under 1 minute.	
Effort and Collaboration	An exceeding amount of time and effort are present and the task responsibilities were shared equitably among group partners.	A substantial amount of effort is present and the task responsibilities were shared equitably among group partners.	An average amount of effort is present, and the task responsibilities were not shared equitably among group partners.	A poor amount of effort is present, and the task responsibilities were not shared equitably among group partners.	
Reflection	Writing is clear, concise, and well organized. Thoughts are expressed in a coherent and logical manner. Contains 2 or more paragraphs with very few grammatical errors present.	Writing is mostly clear, concise, and well organized. Thoughts are expressed in a coherent and logical manner. Contains 1-2 paragraphs with several grammatical errors present.	Writing is unclear and disorganized. Thoughts are not expressed in a logical manner. Contains 1-2 paragraphs with several grammatical errors present.	Writing is unclear and disorganized. Thoughts ramble and make little sense. Contains 1 paragraph with many grammatical errors present.	

STEAM Assessment

Sample 4; (Integrated Strands in relation to the project from integrated subjects)

Unit: (Integrated Units from all Subjects in this project)

Content Standard: (Integrated Content Standard from all Subjects in project)

Benchmark: (Integrated Benchmarks from all Subjects in this project)

Topic: (Integrated Topics from all Subjects in this project)

Lesson topic: (Integrated Topics from all Subjects in concern)

Instructional Objective (s): Students will be able to;

- Create a STEAM project "building a prototype model of a catapult launching system" to enhance their understand of this concept

VASK-MT	
Values/Attitudes	Appreciate the beauty of the application of mathematics during the designing process of the project.
Skills	Calculating size and space Time management and efficiency, Linear measurement and scaling techniques, Calculating mechanical advantage
Knowledge	Size and space Time management and efficiency, Linear measurement and scaling techniques
Mathematical Thinking	Think about how to integrate and apply the mathematical knowledge in the project

What is to be assessed? - (KSAVs)

Integrated subjects concepts used designing the projects.

Purpose of the assessment

To measure students proficiency on the achievement of the benchmarks and learning objectives for integrated subjects in the project. (STEAM Project)

Expected level of proficiency

All students are expected to;

- "Build a prototype model of a catapult launching system" through integrating concepts learned in other subjects.

Performance Task

Student will carry out a project worth 30 marks that should contribute to the School Learning Improvement Program (SLIP). This project will assess students proficiency on the mentioned benchmarks. In order for this assessment type to attain its intended purpose the following must be done carefully;

Task: Students will be given a month to complete this project.

(1) all grade 12 Mathematics teachers discuss the STEAM project with their HOD

(2) the Mathematics HOD brings this project to the attention of the Head Teacher hence it will involve the learning of all grade 12 classes in the school.

(3) once approved by the Head Teacher, the Mathematics HOD now convenes a meeting with all other subject HOD to integrate this project into their learning. HOD for Mathematics will have developed criteria already and will discuss around that.

(4) the HOD for other subjects meet with their respective subject teachers to gauge their views and write up criteria's with reference to the theme of the project, "STEM Design and Engineering Challenge" bringing out the essence of their subjects in this project.

(5) the Head Teacher then convenes a meeting with all teachers as they are now aware of the project. HOD for respective subjects give feedback from their meetings. Issues concerning this project must be ironed out and all subjects now carry out this assessment, starting with Mathematics.

The grade 12 Mathematics teacher will now do the following;

(i) Group the students into groups of 6 to design (drawing and manual) a tangible technology that will enhance the notion of "building a prototype model of a catapult launching system"

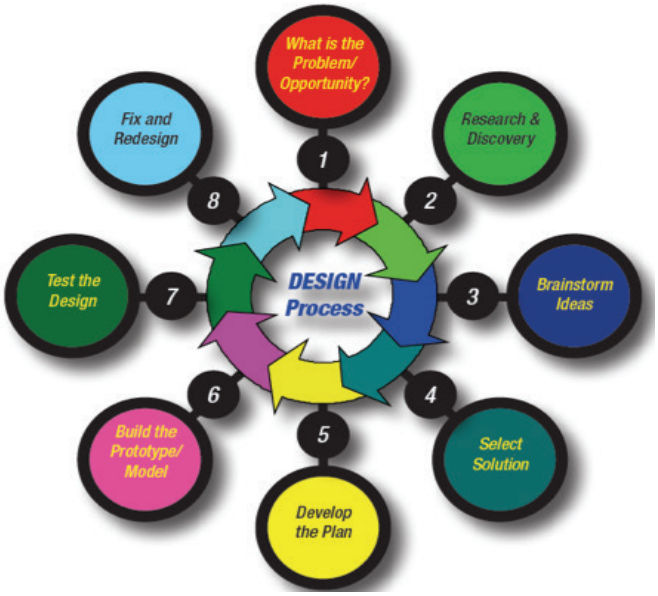
(ii) The teacher then assesses their designs and the best designs now compete with the other best designs from other grade 12 classes.

(iii) All the best designers now create models of their designs with assistance from their class members. At this stage the other subjects now carry forward this assessed projects theme, 'building a prototype model of a catapult launching system" however in the context of their subjects. STEAM is an integrated approach of teaching. All subjects must incorporate the theme put forward by Mathematics. They develop criteria that should address this theme. For instance; Technology and Industrial Arts (TIA) will develop criteria that will engage the students to construct the models. Science teachers will develop criteria to test students' knowledge of the Science process of Engineering Design thinking when they create the models around the theme of "prototype model of a catapult launching system". The English subject teachers will set criteria and guidelines for students on how to write reports so they write to tell others what they have learned and experienced. They must also be given guidelines to writing report. Students get to write report of how they designed this technology. The Mathematics teacher will provide criteria for the students in terms of the measurements, angles and operations used to work out the size and shape of the technology.

Task: Students will be given 6 weeks to complete this project. They are to;

- Design and build a prototype model of a catapult launching system that is easy to use and easy to transport.
- Follow the Design Process to prepare their prototype model in time.
- Write and prepare a short presentation to explain the catapult that was built and the process of building it.





Design Specification:

The catapult should be designed to launch a golf ball at least fifteen feet, to a 18cm x 18cm target.

- The catapult should include a system for determining range, reliability, and accuracy.
- The catapult should be mobile, yet stable. Outriggers or other support systems need to be included to maintain stability when the launcher is used.
- The catapult should be no larger than 30cm long x 30 cm deep x 90cm tall.
- The catapult should feature a locking pin or trigger that activates the catapult to launch.
- Your team should prepare to deliver a presentation about the merits of your catapult model and design.

Assessment Strategy

Design Project will be used to measure student's proficiency.

The students will be reinforced in the following STEAM concepts.

Science

- Applications of simple machines, including wheels and axles, levers, and pulleys
- Balance and equilibrium
- Energy transformations, such as rotary motion to linear motion
- Mechanical advantage

Technology and Engineering

- Prototyping and modelling
- Invention and innovation
- Structural integrity/strength
- Brainstorming and problem solving
- Trial and error engineering concepts

ARTS

- Perspective drawing (3-D)
- Critical Thinking Process
- Applying the Principles of Graphic design
  - balance
  - proximity
  - repetition
  - colour
  - negative/positive space
- Applying creative process

Math

- Calculating size and space
- Time management and efficiency
- Linear measurement and scaling techniques
- Calculating mechanical advantage

Project Rubric

Category	Advanced	Satisfactory	Partial Credit	Unacceptable
	9 -10 points	7- 8 points	1 - 6 points	0 points
Quality/ Workmanship	Maximum effort was put forth to complete the project in a professional manner. Project demonstrates a high degree of quality and attention to detail. Workmanship is excellent.	Some effort was made to complete the project to a level that was sufficient for grading, but does not meet a professional level of quality or appearance. Workmanship is of acceptable quality.	Minimal effort was made to complete the project and the quality and workmanship is sub-par, but still meets the minimal standard.	Little or no effort was made to produce a quality project. Project obviously does not meet minimal standards.
Creativity/ Design	Project reflects many fundamental elements of design and creativity. Project demonstrates an advanced understanding of creative thinking and attention to aesthetics and presentation.	Project reflects some of the elements of design and creativity, but lacks attention to aesthetics and presentation.	Project was completed, but does not reflect the acceptable levels of design and creativity. Effort was minimal and project is mediocre at best.	Project was not completed on time or reflects little or no effort to complete assignment at an acceptable level.
Functionality	Project meets or exceeds the design requirements of purpose and functionality. All elements of the design have been met and the project does what it was designed to do.	Project meets some of the design requirements of purpose and functionality. Not all elements of the design have been met, but the project does what it was designed to do.	Project is somewhat functional, but reflects minimal effort. It is intermittent and doesn't always do what it was designed to do.	Project does not work and demonstrates a lack of effort or understanding of the basic elements of functionality and purpose.
Design Process	Project reflects a clear understanding and application of design process including evidence of research, brainstorming, design and problem solving, prototyping and testing.	Project reflects some understanding and application of accepted design loop principles and sequence including evidence of research, brainstorming, design and problem solving, prototyping and testing.	Project reflects minimal understanding and application of design process.	Project does not show evidence that design process was used. Project does not meet accepted levels of design criteria.
Criteria/ Constraints	Project was completed with all constraints and criteria met or exceeded. Reflects attention to detail and quality.	Project was completed with some of the constraints and criteria met. Reflects some attention to detail, but quality is minimal.	Project was completed with a few of the constraints and criteria met. Reflects minimal effort and lacks detail or quality.	Project was not completed and does not reflect the adherence to the constraints or criteria.

Time Management	Project completed and turned in on time. Student worked diligently when project time was available. Student was on task most of the time.	Project was completed, but had notable errors. Student utilized project time somewhat efficiently, but spent time socializing. Student was on task 70% - 80% of the time.	Project was not turned in on time and/or complete. The student was on task less than 60% of the time.	Project was not turned in on time and was not completed. Student wasted project time and at times was disruptive to others.
Resource Management	Always takes responsibility for use and care of all building components and resources. Always returns building components and materials to proper storage compartments.	Consistently takes responsibility for use and care of building components and resources. Somewhat consistent in returning building components to proper storage compartments.	Sometimes takes responsibility for use and care of building components and resources. Inconsistent in returning building components to proper storage compartments.	Does not take responsibility for the proper use and care of building components and resources. Is careless and does not practice proper storage and safety practices.
Teamwork	Notable teamwork shown with a determination to participate/contribute to team success. Completed required individual tasks that contributed to the success of the team.	Teamwork was noted, but was sometimes off task or working on non-related tasks. Contributed to the success of the team, but could have been more engaged to complete tasks sooner.	Notable time off-task with minimal effort given for team success, or did the project alone without relying on others to do their share of the project.	Was not a team player. Either took over project completely, or did not engage in team direction or plans.
Writing/ Reflection	Writing/reflection is very well organized and explained. Student includes all details in design process. Document has almost no grammatical errors.	Writing/reflection is somewhat organized and explained. Student includes most details in design process. Document has very few grammatical errors.	Writing/reflection is not organized and explained. Student includes only a few details in design process. Document has many grammatical errors.	Writing/reflection is incomplete or not turned in. Student includes no details in design process. Document has many grammatical errors.
Presentation	Presentation was well organized and presented in a logical sequence. Presentation reflects a full knowledge of the topic with clear answers and explanations to questions asked.	Presentation was fairly organized and most information presented in a logical sequence. Answers to questions were vague or lacked clarity or accuracy.	Presentation was unorganized and lacked a logical sequence. Presentation reflected little attention to detail. Answers to questions were inaccurate and confusing.	Presentation was not acceptable and reflects a lack of organization or knowledge of the topic. Presentation shows little effort to meet expectations.

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Appendices

APPENDIX 1: BLOOM'S TAXONOMY

LEVEL OF UNDERSTANDING	KEY VERBS
CREATING **Can the student create a new product or point of view?**	Construct, design, and develop, generate, hypothesize, invent, plan, produce, compose, create, make, perform, plan, produce, assemble, formulate,
EVALUATING **Can the student justify a stand or decision?**	Appraise, argue, assess, choose, conclude, critique, decide, defend, evaluate, judge, justify, predict, prioritize, provoke, rank, rate, select, support, monitor,
ANALYSING **Can the student distinguish between the different parts?**	Analysing, characterize, classify, compare, contrast, debate, criticise, deconstruct, deduce, differentiate, discriminate, distinguish, examine, organize, outline, relate, research, separate, experiment, question, test,
APPLYING **Can the student use the information in a new way**	Apply, change, choose, compute, dramatize, implement, interview, prepare, produce, role play, select, show, transfer, use, demonstrate, illustrate, interpret, operate, sketch, solve, write,
UNDERSTANDING **Can the student comprehend ideas or concepts?**	Classify, compare, exemplify, conclude, demonstrate, discuss, explain, identify, illustrate, interpret, paraphrase, predict, report, translate, describe, classify,
REMEMBERING **Can the student recall or remember the information?**	Define, describe, draw, find, identify, label, list, match, name, quote, recall, recite, tell, write, duplicate, memorise, recall, repeat, reproduce, state,

APPENDIX 2: 21ST CENTURY SKILLS

WAYS OF THINKING	**Creativity and innovation** • Think creatively • Work creatively with others • Implement innovations **Critical thinking, problem solving and decision making** • Reason effectively and evaluate evidence • Solve problems • Articulate findings **Learning to learn and meta-cognition** • Self-motivation • Positive appreciation of learning • Adaptability and flexibility
WAYS OF WORKING	**Communication** • Competency in written and oral language • Open minded and preparedness to listen • Sensitivity to cultural differences **Collaboration and teamwork** • Interact effectively with others • Work effectively in diverse teams • Prioritise, plan and manage projects
TOOLS FOR WORKING	**Information literacy** • Access and evaluate information • Use and manage information • Apply technology effectively **ICT literacy** • Open to new ideas, information, tools and ways of thinking • Use ICT accurately, creatively, ethically and legally • Be aware of cultural and social differences • Apply technology appropriately and effectively
LIVING IN THE WORLD	**Citizenship – global and local** • Awareness and understanding of rights and responsibilities as a global citizen • Preparedness to participate in community activities • Respect the values and privacy of others **Personal and social responsibility** • Communicate constructively in different social situations • Understand different viewpoints and perspectives **Life and career** • Adapt to change • Manage goals and time • Be a self-directed learner • Interact effectively with others

APPENDIX 3: TEACHING AND LEARNING STRATEGIES

STRATEGY	TEACHER	STUDENTS
CASE STUDY **Used to extend students' understanding of real life issues**	Provide students with case studies related to the topic of the lesson and allow them to analyse and evaluate.	Study the case study and identify the problem addressed. They analyse the problem and suggest solutions supported by conceptual justifications and make presentations. This enriches the students' existing knowledge of the topic.
DEBATE **A method used to increase students' interest, involvement and participation**	Provide the topic or question of debate on current issues affecting a bigger population, clearly outlining the expectations of the debate. Explain the steps involved in debating and set a criteria/standard to be achieved.	Conduct researches to gather supporting evidence about the selected topic and summarising the points. They are engaged in collaborative learning by delegating and sharing tasks to group members. When debating, they improve their communication skills.
DISCUSSION **The purpose of discussion is to educate students about the process of group thinking and collective decision.**	The teacher opens a discussion on certain topic by asking essential questions. During the discussion, the teacher reinforces and emphasises on important points from students responses. Teacher guide the direction to motivate students to explore the topic in greater depth and the topic in more detail. Use how and why follow-up questions to guide the discussion toward the objective of helping students understand the subject and summarise main ideas.	Students ponder over the question and answer by providing ideas, experiences and examples. Students participate in the discussion by exchanging ideas with others.
GAMES AND SIMULATIONS **Encourages motivation and creates a spirit of competition and challenge to enhance learning**	Being creative and select appropriate games for the topic of the lesson. Give clear instructions and guidelines. The game selected must be fun and build a competitive spirit to score more than their peers to win small prices.	Go into groups and organize. Follow the instructions and play to win
OBSERVATION **Method used to allow students to work independently to discover why and how things happen as the way they are. It builds curiosity.**	Give instructions and monitor every activity students do	Students possess instinct of curiosity and are curious to see the things for themselves and particularly those things which exist around them. A thing observed and a fact discovered by the child for himself becomes a part of mental life of the child. It is certainly more valuable to him than the same fact or facts learnt from the teacher or a book. Students Observe and ask essential questions Record Interpret

PEER TEACHING & LEARNING *(power point presentations, pair learning)* Students teach each other using different ways to learn from each other. It encourages; team work, develops confidence, feel free to ask questions, improves communication skills and most importantly develop the spirit of inquiry.	Distribute topics to groups to research and teach others in the classroom. Go through the basics of how to present their peer teaching.	Go into their established working groups. Develop a plan for the topic. Each group member is allocated a task to work on. Research and collect information about the topic allocated to the group. Outline the important points from the research and present their findings in class.
PERFORMANCE-RELATED TASKS (dramatization, song/ lyrics, wall magazines) Encourages creativity and take on the overarching ideas of the topic and are able to recall them at a later date	Students are given the opportunity to perform the using the main ideas of a topic. Provide the guidelines, expectations and the set criteria	Go into their established working groups. Being creative and create dramas, songs/ lyrics or wall magazines in line with the topic.
PROJECT (individual/group) Helps students complete tasks individually or collectively	Teacher outline the steps and procedures of how to do and the criteria	Students are involved in investigations and finding solutions to problems to real life experiences. They carry out researches to analyse the causes and effects of problems to provide achievable solutions. Students carefully utilise the problem-solving approach to complete projects.
USE MEDIA & TECHNOLOGY to teach and generate engagement depending on the age of the students	Show a full movie, an animated one, a few episodes form documentaries, you tube movies and others depending on the lesson. Provide questions for students to answer before viewing	Viewing can provoke questions, debates, critical thinking, emotion and reaction. After viewing, students engage in critical thinking and debate

APPENDIX 4 : ASSESSMENT STRATEGIES

STRATEGY	DESCRIPTION
ANALOGIES	Students create an analogy between something they are familiar with and the new information they have learned. When asking students to explain the analogy, it will show the depth of their understanding of a topic.
CLASSROOM PRESENTATIONS	A classroom presentation is an assessment strategy that requires students to verbalize their knowledge, select and present samples of finished work, and organize their thoughts about a topic in order to present a summary of their learning. It may provide the basis for assessment upon completion of a student's project or essay.
CONFERENCES	A conference is a formal or informal meeting between the teacher and a student for the purpose of exchanging information or sharing ideas. A conference might be held to explore the student's thinking and suggest next steps; assess the student's level of understanding of a particular concept or procedure; and review, clarify, and extend what the student has already completed
DISCUSSIONS	Having a class discussion on a unit of study provides teachers with valuable information about what the students know about the subject. Focus the discussions on higher level thinking skills and allow students to reflect their learning before the discussion commences.
ESSAYS	An essay is a writing sample in which a student constructs a response to a question, topic, or brief statement, and supplies supporting details or arguments. The essay allows the teacher to assess the student's understanding and/or ability to analyse and synthesize information.
EXHIBITIONS/ DEMONSTRATIONS	An exhibition/demonstration is a performance in a public setting, during which a student explains and applies a process, procedure, etc., in concrete ways to show individual achievement of specific skills and knowledge.
INTERVIEWS	An interview is a face-to-face conversation in which teacher and student use inquiry to share their knowledge and understanding of a topic or problem, and can be used by the teacher to explore the student's thinking; assess the student's level of understanding of a concept or procedure and gather information, obtain clarification, determine positions, and probe for motivations.
LEARNING LOGS	A learning log is an ongoing, visible record kept by a student and recording what he or she is doing or thinking while working on a particular task or assignment. It can be used to assess student progress and growth over time.
OBSERVATION	Observation is a process of systematically viewing and recording students while they work, for the purpose of making programming and instruction decisions. Observation can take place at any time and in any setting. It provides information on students' strengths and weaknesses, learning styles, interests, and attitudes.
PEER ASSESSMENT	Assessment by peers is a powerful way to gather information about students and their understanding. Students can use set criteria to assess the work of their classmates.
PERFORMANCE TASKS	During a performance task, students create, produce, perform, or present works on "real world" issues. The performance task may be used to assess a skill or proficiency, and provides useful information on the process as well as the product.
PORTFOLIOS	A portfolio is a collection of samples of a student's work, and is focused, selective, reflective, and collaborative. It offers a visual demonstration of a student's achievement, capabilities, strengths, weaknesses, knowledge, and specific skills, over time and in a variety of contexts.

QUESTIONS AND ANSWERS (ORAL)	In the question–and-answer strategy, the teacher poses a question and the student answers verbally, rather than in writing. This strategy helps the teacher to determine whether students understand what is being, or has been, presented, and helps students to extend their thinking, generate ideas, or solve problems.
QUIZZES, TESTS, EXAM-INATIONS	A quiz, test, or examination requires students to respond to prompts in order to demonstrate their knowledge (orally or in writing) or their skills (e.g., through performance). Quizzes are usually short; examinations are usually longer. Quizzes, tests, or examinations can be adapted for exceptional students and for re-teaching and retesting.
QUESTIONNAIRES	Questionnaires can be used for a variety of purposes. When used as a formative assessment strategy, they provide teachers with information on student learning that they can use to plan further instruction.
RESPONSE JOURNALS	A response journal is a student's personal record containing written, reflective responses to material he or she is reading, viewing, listening to, or discussing. The response journal can be used as an assessment tool in all subject areas.
SELECTED RESPONSES	Strictly speaking a part of quizzes, tests, and examinations, selected responses require students to identify the one correct answer. The strategy can take the form of multiple-choice or true/false formats. Selected response is a commonly used formal procedure for gathering objective evidence about student learning, specifically in memory, recall, and comprehension.
STUDENT SELF-ASSESSMENTS	Self-assessment is a process by which the student gathers information about, and reflects on, his or her own learning. It is the student's own assessment of personal progress in terms of knowledge, skills, processes, or attitudes. Self-assessment leads students to a greater awareness and understanding of themselves as learners.

APPENDIX 5: Standard-based Lesson Plan Template

Strand: .....

Unit: .....

Content Standard: .....

Benchmark: .....

Topic : .....

Lesson Title: .....

Lesson Objective (s): By the end of the lesson, students will be able to;

- .....
- .....

Materials:

Key Concepts(ASK-MT)	
Attitudes / Values	
Skills	
Knowledge	
Mathematics Thinking	

Lesson Procedure

Teacher Activity	Student Activity
Introduction	(time in minutes)
Body	(time in minutes)
Modeling	
Guided Practice	
Independent Practice	
Conclusion	(time in minutes)

Assessment/Lesson Evaluation

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APPENDIX 6: Standard based Lesson Plan template - Integrating STEAM

Strand: .....

Unit: .....

Content Standard: .....

Benchmark: .....

Topic : .....

Lesson Title: .....

Lesson Objective (s): By the end of the lesson, students will be able to;

- .....

Essential Questions

- .....

Materials:

Key Concepts(ASK-MT)	
Attitudes / Values	
Skills	
Knowledge	
Mathematics Thinking	
STEAM Knowledge and Skills	
Skills	
Knowledge	
STEAM Performance Indicator:	

Lesson Procedure

Teacher Activity	Student Activity
Introduction	(time in minutes)
Body	(time in minutes)
Modeling	
Guided Practice	
Independent Practice	
Conclusion	(time in minutes)

Assessment/Lesson Evaluation

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APPENDIX 7: Time Allocation

Grade 9 and 10	No. lesson/ wk	Min/week	Grade 11 and 12	No. lessons/ wk	Min/week
English	6	6 x 40 = 240	Applied English	6	6 x 40 = 240
Mathematics	5	5 x 40 = 200	L & L	6	6 x 40 = 240
Science	5	5 x 40 = 200	**Advanced Math**	**5**	**5 x 80 = 400**
Social Science	5	5 x 40 = 200	General Math	6	8 x 40 = 320
PD	5	5 x 40 = 200	Physics	6	6 x 40 = 240
Business Studies	5	5 x 40 = 200	Biology	6	6 x 40 = 240
Design & Technology	5	5 x 40 = 200	Chemistry	6	6 x 40 = 240
Arts	5	5 x 40 = 200	Applied Science	6	6 x 40 = 240
CCVE	3	3 x 40 = 120	Geology	6	6 x 40 = 240
RI	1	1 x 60 = 60	Geography	6	6 x 40 = 240
Agriculture	5	5 x 40 = 200	History	6	6 x 40 = 240
TOTALS	50	2020 min/wk	Legal Studies	6	6 x 40 = 240
			HPE	6	6 x 40 = 240
			PE	6	6 x 40 = 240
			RE	1	1 x 60 = 60
			Business Studies	6	6 x 40 = 240
			Accounting	6	6 x 40 = 240
			Economics	6	6 x 40 = 240
			Design & Tech	6	6 x 40 = 240
			Computer Studies	6	6 x 40 = 240
			ICT	6	6 x 40 = 240
			CCVE	2	3 x 40 = 120
			ANRM	6	6 x 40 = 240
			TOTALS	128 lessons/ wk	5,460 min/wk

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